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Aim and Scope

International Journal of Neutrosophic Science (IJNS) is a peer-review journal publishing high quality experimental and theoretical research in all areas of Neutrosophic and its Applications. IJNS is published quarterly. IJNS is devoted to the publication of peer-reviewed original research papers lying in the domain of neutrosophic sets and systems. Papers submitted for possible publication may concern with foundations, neutrosophic logic and mathematical structures in the neutrosophic setting. Besides providing emphasis on topics like artificial intelligence, pattern recognition, image processing, robotics, decision making, data analysis, data mining, applications of neutrosophic mathematical theories contributing to economics, finance, management, industries, electronics, and communications are promoted. Variants of neutrosophic sets including refined neutrosophic set (RNS). Articles evolving algorithms making computational work handy are welcome.

Topics of Interest

IJNS promotes research and reflects the most recent advances of neutrosophic Sciences in diverse disciplines, with emphasis on the following aspects, but certainly not limited to:

- Neutrosophic sets
- Neutrosophic algebra
- Neutrosophic topolog
- Neutrosophic graphs
- Neutrosophic probabilities
- Neutrosophic tools for decision making
- Neutrosophic theory for machine learning
- Neutrosophic statistics
- Neutrosophic numerical measures
- Classical neutrosophic numbers
- A neutrosophic hypothesis
- The neutrosophic level of significance
- The neutrosophic confidence interval
- The neutrosophic central limit theorem
- Neutrosophic theory in bioinformatics and medical analytics
- Neutrosophic tools for big data analytics
- Neutrosophic tools for deep learning
- Neutrosophic tools for data visualization
- Quadripartitioned single-valued neutrosophic sets
- Refined single-valued neutrosophic sets
- Applications of neutrosophic logic in image processing
- Neutrosophic logic for feature learning, classification, regression, and clustering

- Neutrosophic knowledge retrieval of medical images
- Neutrosophic set theory for large-scale image and multimedia processing
- Neutrosophic set theory for brain-machine interfaces and medical signal analysis
- Applications of neutrosophic theory in large-scale healthcare data
- Neutrosophic set-based multimodal sensor data
- Neutrosophic set-based array processing and analysis
- Wireless sensor networks Neutrosophic set-based Crowd-sourcing
- Neutrosophic set-based heterogeneous data mining
- Neutrosophic in Virtual Reality
- Neutrosophic and Plithogenic theories in Humanities and Social Sciences
- Neutrosophic and Plithogenic theories in decision making
- Neutrosophic in Astronomy and Space Sciences

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True-False Set is a particular case of the Refined Neutrosophic Set

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Abstract

Borzooei, Mohseni Takallo, and Jun recently proposed a new type of set, called True-False Set [1], and they claimed it is a generalization of Neutrosophic Set [2]. We prove that this assertion is untrue. Actually it's the opposite, the True-False Set is a particular case of the Refined Neutrosophic Set..

Keywords: Refined Neutrosophic Set, True-False Set, Neutrosophic Set, Indeterminacy.

1. Definition of True-False Set [1]

A True-False set (TF-set), on a none-empty set X, is a structure of the form:

$A_{TFS} = \{x;\ t_A(x), T_A(x), f_A(x), F_A(x) | x \in X\}$; the index "TFS" stands for True-False Set;

where $t_A: X \to [0, 1]$; t_A represents the single-valued truth function;

$T_A: X \to I([0, 1])$, where $I([0, 1])$ is the set of all subintervals of $[0, 1]$; T_A represents the interval-valued truth function;

$f_A: X \to [0, 1]$; f_A represents the single-valued falsehood function;

$F_A: X \to I([0, 1])$; F_A represents the interval-valued falsehood function.

It is not clear why two truth-functions and two falsehood-functions are needed for the same element x. There is no justification.

2. Definition of neutrosophic set [2]

We try to use similar notations and language in order to make easy comparison between the two types of sets.

Let X be a non-empty universe of discourse.

A Neutrosophic Set on X is a structure of the form:

$A_{NS} = \{x;\ T_A(x), I_A(x), F_A(x) | x \in X\}$,

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where $T_A, I_A, F_A: X \to \mathcal{P}(]^-0, 1^+[)$, where $\mathcal{P}(]^-0, 1^+[)$ is the set of all standard or nonstandard subsets of the non-standard interval $]^-0, 1^+[$.

3. Distinctions between True-False Set and Neutrosophic Set

1) Clearly $\mathcal{P}(]^-0, 1^+[) \supsetneq I([0, 1])$. From this point of view, the neutrosophic set is larger than the True-False Set.

$\mathcal{P}(]^-0, 1^+[)$ includes not only strandard subintervals of $[0, 1]$ as $I([0, 1])$, but any standard subsets of $[0, 1]$.

2) $\mathcal{P}(]^-0, 1^+[)$ also includes non-standard subsets of $]^-0, 1^+[$, left and right monads, binads from non-standard analysis, that help make a distinction between absolute truth (truth in all possible worlds, according to Leibniz), whose truth-value is $T_A(x) = 1^+$, where $1^+ = 1 + \varepsilon > 1$ and ε is a positive infinitesimal number.

Similarly for absolute / relative indeterminacy and respectively falsehood.

The True-False Set cannot make distinctions between absolute and relative truth/falsehood.

3) Neutrosophic Set is much more complex as structure than the True-False Set; Neutrosophic Set has been further extended Neutrosophic Overset (where the neutrosophic components could be > 1), Neutrosophic Underset (where the neutrosophic components could be < 0), and Neutrosophic Offset (where the neutrosophic components could be > 1 and < 0) in 2007 & 2016 ([3], [4]).

4. Transformation of a Single-Valued Neutrosophic Set to a True-False Set [1]

The authors of [1] considered only the simplest form of the Neutrosophic Set, i.e. when the neutrosophic components T, I, F are single (crisp) numbers in [0, 1], while the general definition [2] of neutrosophic set stated since 1998 that T, I, F can be any subsets of [0, 1], or any nonstandard subsets of the non-standard unit interval $]^-0, 1^+[$.

They considered the single-valued neutrosophic set:

$$A_{NS} = \{x;\ T_{NS}(x), I_{NS}(x), F_{NS}(x) | x \in X\},$$

where T_{NS}, I_{NS}, $F_{NS}: X \to [0, 1]$ are single-valued truth, indeterminacy, and falsehood functions respectively. The index “NS” stands for Neutrosophic Set (we adjusted their Greek letter notations to Latin ones, in order to exactly match the common use notations of the neutrosophic set).

They transformed it to a True-False Set in the following way:

$$t(x) = T_{NS}(x);$$

$$f(x) = F_{NS}(x);$$

$$T_{TFS}(x) = \begin{cases} [T_{NS}(x), I_{NS}(x)], \text{if } T_{NS}(x) \le I_{NS}(x); \\ [I_{NS}(x), T_{NS}(x)], \text{if } I_{NS}(x) \le T_{NS}(x); \end{cases}$$

$$F_{TFS}(x) = \begin{cases} [F_{NS}(x), I_{NS}(x)], \text{if } F_{NS}(x) \le I_{NS}(x); \\ [I_{NS}(x), F_{NS}(x)], \text{if } I_{NS}(x) \le F_{NS}(x). \end{cases}$$

And they formed the following True-False Set:

$$A_{TFS} = \{x;\ t(x), T_{TFS}(x), f(x), F_{TFS}(x) | x \in X\} = \{\langle x;\ T_{NS}(x), T_{TFS}(x), F_{NS}(x), F_{TFS}(x)\rangle | x \in X\}.$$

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This True-False Set, A_{TFS}, has two truth-functions and two-falsehood functions, but no indeterminacy (neutrality) function (they removed it).

Transforming the neutrosophic set A_{NS} into a true-false set A_{TFS} is just a mathematical artifact. It is not proven that A_{NS} is equivalent to A_{TFS}. Actually, we'll prove below that they are not.

Other mathematical transformations can be designed as well, constructing new intervals, or combining the neutrosophic functions in other ways, etc. But the equivalence, if any, should be proven.

5. Indeterminacy (Neutrality)

The indeterminacy (neutrality) is the quintessence (the flavor) of neutrosophic set, that stringently distinguishes it from previous types of sets.

By eliminating the indeterminacy (or neutrality) from the neutrosophic set A_{NS}, when constructing a true-false set A_{TFS}, the true-false set A_{TFS} becomes defficient, incapable of characterizing the neutrosophic triads of the form $(\langle A\rangle, \langle \text{neut}A\rangle, \langle \text{anti}A\rangle)$, where $\langle A\rangle$ is an item (idea, proposition, attribute, concept, etc.), $\langle \text{anti}A\rangle$ is its opposite, and $\langle \text{neut}A\rangle$ is the neutral between these opposites.

For example, in games we have such triads (where $\langle A\rangle = $ winning): <winning, tie game, loosing>.

6. Numerical Counter-Example of Transforming a Single-Valued Neutrosophic set to a True-False Set

Let's take only one element from a single valued neutrosophic set (for the other elements it will be similar):

$x_{NS}(0.3, 0.4, 0.2)$, hence $T_{NS}(x) = 0.3$, $I_{NS}(x) = 0.4$, $F_{NS}(x) = 0.2$.

Let's transform it into a true-false set's element according to [1]:

$x_{TFS}(0.3, [0.3, 0.4], 0.2, [0.2, 0.4])$, hence $t_{TFS}(x) = 0.3$, $T_{TFS}(x) = [0.3, 0.4]$, $f_{TFS}(x) = 0.2$, $F_{TFS}(x) = [0.2, 0.4]$.

The indeterminacy $I_{NS}(x) = 0.4$ into the neutrosophic set has been replaced into the true-false set by an interval-value truth $T_{TFS}(x) = [0.3, 0.4]$ and an interval-value falsehood $F_{TFS}(x) = [0.2, 0.4]$. But these are a totally different results.

If, with respect to an element, the indeterminacy-membership is 0.4, this is not equivalent with element's truth-membership be equal to [0.3, 0.4] and its false-membership be equal to [0.2, 0.4].

7. Other Counter-Examples

Let $x_{NS}(0.3, 0.4, 0.2)$ represent, with respect to the player x in a game where he plays against others, that his degree of winning ($T_{NS} = 0.3$), his degree of tie game ($I_{NS} = 0.4$), and his degree of loosing ($F_{NS} = 0.2$).

By transforming x_{NS} to $x_{TFS}(0.3, [0.3, 0.4], 0.2, [0.2, 0.4])$, we get that with respect to the same player x, his degree of winning is 0.3 or $[0.3, 0.4]$, and his degree of loosing is 0.2 or $[0.2, 0.4]$.

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7.1. Therefore, the true-false set does not provide any degree of “tie game”, so this type of set is incomplete. The true-false set does not catch the middle side (neutrality, or indeterminacy) in between opposites.

7.2. Another drawback is that TFS increases the imprecision of the truth function: for $T_{NS} = 0.3$, it gets $T_{TFS} = 0.3$ or $[0.3, 0.4]$, so the truth value becomes vaguer after the TFS transformation.

TFS increases the imprecision of the falsehood function as well: for FNS = 0.2, it gets FTFS = 0.2 or [0.2, 0.4], so the falsehood value becomes vaguer after the TFS transformation.

8. The True-False Set is a particular case of the Refined Neutrosophic Set

In the Refined Neutrosophic Set (Logic, Probability), T can be split into subcomponents $T_1, T_2, ..., T_p$, and I into $I_1, I_2, ..., I_r$, and F into $F_1, F_2, ...,F_s$, where $p, r, s \in \{0, 1, 2, \dots, \infty\}$ and $p + r + s = n \in \{0, 1, 2, \dots, \infty\}$. By index = 0, of a neutrosophic component T, I, or F, or any of their subcomponents, we denote the empty set, i.e. $T_0 = \phi$, $I_0 = \phi$, $F_0 = \phi$. The case (T_0, I_0, F_0) is the most degenerated one. See [4].

From (T, I, F), where T, I, F are any subsets of [0, 1], we replace $I_0 = \phi$ (empty set), and refine/split T into T_1 (single-valued truth component) and T_2 (as an interval-valued truth component), while F is similarly refined/split into F_1 (as a single-valued falsehood component) and F_2 (as an interval-valued falsehood component). Therefore, we replaced $p = 2$, $r = 0$, and $s = 2$ into the general form of the Refined Neutrosophic Set, and we found the True-False Set $(T_1, T_2, I_0 = \phi, F_1, F_2)$.

9. Conclusion

We proved that the transformation of the Neutrosophic Set into a True-False Set does not give equivalent results by using several coounter-examples. Also, we proved that the True-False Set is a particular case of Refined Neutrosophic Set.

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An Introduction to Neutro-Fine Topology with Separation Axioms and Decision Making

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Abstract

The set which describes the uncertainty incident with three levels of attributes is entitled as a neutrosophic set. The unique collection of open sets which contains all types of open sets is termed as fine-open sets. The current study introduces a topology on merging these two sets, called neutro-fine topological space. Additionally, the approach of separation axioms is implemented in such space. Furthermore, the real-life application is examined as a decision-making problem in this space. The problem is to make an unfavorable query into a favorable one by determining the complement and absolute complement of such issued neutro-fine open sets. This problem desires to find a positive solution. The solving stepwise mechanism reveals in the algorithm, also formulae provide to compute the outcome with explanatory examples.

Keywords: Subset of neutrosophic sets, neutro-fine sets (NFSs), neutro-fine topological space (NFTS), neutro-fine open sets (NFOSs), neutro-fine interior and closure, neutro-fine $T_{i=0,1,2}$-spaces, absolute complement, decision making (DM).

1. Introduction

The fuzzy set (FS) is an advanced version of the classical set. FS was introduced in (1965) by Zadeh [32], whose elements describe vague features as true and false membership functions. The FS theory applied in the boundless area of a domain, while in (1986) this theory was extended as an intuitionistic fuzzy set (IFS) theory by Atanassov [6]. Later, in 1998 Smarandache [26] explored a neutrosophic set (NS) that contains one more membership function called indeterminacy degrees. Also, he widespreaded the NS on IFS [27] and newly projected his work on features valued set, called plithogenic set (PS) [28]. Nowadays, these sets made an outstanding impact on many applications [1, 2, 11, 18, 29] and plays vital role in COVID-19 [25], decision making (DM) problems [3, 4, 5] and multi-criteria DM (MCDM) problems [10, 21].

Topology is a study of flexible objects under frequent damages without splitting. In recent times, topological space (TS) is performing a lead character in the enormous branch of applied sciences and numerous categories of mathematics. The topological structure developed on NS as a generalization of IFTS which was originated in (2012) by Salama & Alblowi [23, 24], named as neutrosophic topological space (NTS). Few typical sets, open sets, and other TS explored [8, 9, 14, 15, 20, 30], and extended to separation axioms [7, 31] on such TS.

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The most general class of sets which contains few open sets termed as fine-open sets (FOSs), in (2012) by Powar & Rajak [16], and investigated the special case of generalized TS, called fine- topological space (FTS). Many researchers studied this concept on some sets like FS [13, 22] and others [12, 19, 17].

This paper desires to initiate the new form of TS to put together NS and FS, through defining the concept of the subset of NSs on these TS. The notion of the interior and closure are launched and certain theorems are stated with proof, also disproved in counter examples. The idea of separation axioms is also executed in NFTS. Moreover, the DM problem describes the negative state of the problem into a positive solution by determining the absolute complement of each NFOS. The procedure of this problem-solving method is listed in the algorithm and a unique decision is computed with the specified formula.

The layout of this study are arranged as follows. In Section 2, essential definitions of NS and FOS are recollected. In Section 3, the subset of NS, NFS, NFTS, interior, and closure of NFTS are defined and investigated its properties with illustrative examples. In Section 4, the correlation of neutro-fine $T_{i=0,1,2}$ -spaces are explored via perfect examples. In Section 5, the real-life application is intimated to take a correct decision on DM problems and an example is investigated in two different manners. At the end of Section 6, the conclusions are conveyed with future works.

2. Fundamental concepts

In this section, some essential definitions associated with this work are pointed.

Definition 2.1 [27] Let W be a non-empty set and $w \in W$. A NS R in W is characterized by a truth-membership function T_R , an indeterminacy-membership function I_R , and a false-membership function F_R which are subsets of $]^{-}0,1^{+}[$ and is defined as

$$R = \{\langle w, T_R(w), I_R(w), F_R(w)\rangle : w \in W\},$$

where

$$0 \leq \sup T_R(w) + \sup I_R(w) + \sup F_R(w) \leq 3 .$$

Definition 2.2 [31] Let $NS(W)$ be the family of all NSs over the universe W and $w \in W$. Then NS is said to be a neutrosophic point (NP), for $0 \leq \alpha, \beta, \gamma \leq 1$ and is defined as follows:

$$w^{\langle \alpha,\beta,\gamma\rangle}(v) = \begin{cases} (\alpha,\beta,\gamma), & if \quad w = v \\ (0,0,1), & if \quad w \neq v \end{cases}.$$

Every NS is the union of its NPs.

Example 2.3 [31] Let $W = \{w_1, w_2, w_3\}$. Then NS

$$R = \{\langle w_1, .2, .4, .7\rangle, \langle w_2, .6, .3, .1\rangle, \langle w_3, .4, .5, .6\rangle\}$$

is the union of NPs $w_1^{\langle .2,.4,.7\rangle}$, $w_2^{\langle .6,.3,.1\rangle}$ and $w_3^{\langle .4,.5,.6\rangle}$.

Definition 2.4 [23] Let $NS(W)$ denote the family of all NSs over W and $\tau_n \subset NS(W)$. Then τ_η is called a neutrosophic topology (NT) on W if it satisfies the following conditions

(i) $0_n, 1_n \in \tau_n$, where null NS $0_n = \{\langle w, 0, 0, 1)\rangle : w \in W\}$ and an absolute NS $1_n = \{\langle w, 1, 1, 0)\rangle : w \in W\}$.

(ii) the intersection of any finite number of members of τ_n belongs to τ_n .

(iii) the union of any collection of members of τ_n belongs to τ_n .

Then the pair (W, τ_n) is called a NTS.

Every member of τ_n is called τ_n -open neutrosophic set (τ_n - ONS). An NS is called τ_n -closed (τ_n - CNS) if and only if its complement is τ_n - ONS.

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Definition 2.5 [23] Let R be a NS over W. Then the complement of R is denoted by R' and defined by

$$R' = \{\langle w, F_R(w), 1 - I_R(w), T_R(w)\rangle : w \in W\}.$$

Clearly, $(R')' = R$.

Example 2.6 [23] Let $W = \{w_1, w_2, w_3\}$ and $\tau_n = \{0_n, 1_n, R, S, T, U\}$ where R, S, T, and U are NSs over W and are defined as follows

$$R = \{\langle w_1, .2, .4, .7\rangle, \langle w_2, .6, .3, .1\rangle, \langle w_3, .4, .5, .6\rangle\},$$
$$S = \{\langle w_1, .9, .3, .6\rangle, \langle w_2, .6, .5, .4\rangle, \langle w_3, .7, .8, .1\rangle\},$$
$$T = \{\langle w_1, .9, .4, .6\rangle, \langle w_2, .6, .5, .1\rangle, \langle w_3, .7, .8, .1\rangle\} \text{ and}$$
$$U = \{\langle w_1, .2, .3, .7\rangle, \langle w_2, .6, .3, .4\rangle, \langle w_3, .4, .5, .6\rangle\}.$$

Here $R \bigcup S = T$, $R \bigcup T = T$, $R \bigcup U = R$, $S \bigcup T = T$, $S \bigcup U = S$, $T \bigcup U = T$ and $R \bigcap S = U$, $R \bigcap T = R$, $R \bigcap U = U$, $S \bigcap T = S$, $S \bigcap U = U$, $T \bigcap U = U$.

Then R, S, T and U are τ_n - ONSs over W.

Thus (W, τ_n) is a NTS over W.

The complement of τ_n - ONSs are

$$R' = \{\langle w_1, .7, .6, .2\rangle, \langle w_2, .1, .7, .6\rangle, \langle w_3, .6, .5, .4\rangle\},$$
$$S' = \{\langle w_1, .6, .7, .9\rangle, \langle w_2, .4, .5, .6\rangle, \langle w_3, .1, .2, .7\rangle\},$$
$$T' = \{\langle w_1, .6, .6, .9\rangle, \langle w_2, .1, .5, .6\rangle, \langle w_3, .1, .2, .7\rangle\} \text{ and}$$
$$U' = \{\langle w_1, .7, .7, .2\rangle, \langle w_2, .4, .7, .6\rangle, \langle w_3, .6, .5, .4\rangle\}.$$

Then R', S', T' and U' are τ_n - CNSs over W.

Definition 2.7 [16] Let (W, τ) be a topological space and define

$$\varsigma(R_i) = \varsigma_i = \{K_i(\neq W) : K_i \subset W, R_i \bigcap K_i \neq \varphi, \text{for } R_i \in \tau \text{ and } R_i \neq \varphi, W, \text{for some } i \in I, \text{where I is the index set}\}.$$

Now, define $\tau_f = \left\{\phi, W, \bigcup_{i \in I} \varsigma_i\right\}$.

This collection τ_f of subsets of W is called the fine collection of subsets of W and (W, τ, τ_f) is said to be the fine space W generated by the topology τ on W.

Definition 2.8 [16] A subset U of a fine space W is said to be fine-open sets of W if U belongs to the collection τ_f and the complement of every fine-open set of W is called the fine-closed sets of W and denote the collection by F_f.

Example 2.9 [16] Let $W = \{w_1, w_2, w_3\}$ with topology $\tau = \{\phi, W, \{w_1\}\}$.

Clearly, (W, τ) is a topological space over W.

Then (W, τ, τ_f) is a fine-topological space over W,

where the members in

$$\tau_f = \{\phi, W, \{w_1\}, \{w_1, w_2\}, \{w_1, w_3\}\}$$

are fine-open sets, and in

$$F_f = \{\phi, W, \{w_2, w_3\}, \{w_3\}, \{w_2\}\}$$

are fine-closed sets.

Example 2.10 [16] Let $W = \{w_1, w_2, w_3\}$ with topology $\tau = \{\phi, W, \{w_1\}, \{w_1, w_3\}\}$.

Clearly, (W, τ) is a topological space over W.

Then the collection of fine-open sets is

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$$\tau_f = \{\phi, W, \{w_1\}, \{w_1, w_2\}, \{w_1, w_3\}, \{w_3\}, \{w_3, w_2\}\}$$

and the collection of fine-closed sets is

$$F_f = \{\phi, W, \{w_2, w_3\}, \{w_3\}, \{w_2\}, \{w_1, w_2\}, \{w_1\}\}.$$

Thus (W, τ, τ_f) is a fine-topological space over W.

Here $\{w_1, w_2\}, \{w_3, w_2\} \in \tau_f$ but $\{w_1, w_2\} \cap \{w_3, w_2\} = \{w_2\} \notin \tau_f$.

Also, $\{w_1\}, \{w_3\} \in F_f$ but $\{w_1\} \cup \{w_3\} = \{w_1, w_3\} \notin F_f$.

Hence (W, τ_f) and (W, F_f) are not topological space over W.

3. Neutro-Fine Topology

In this section, the conception of NFTS is defined and probable results are carried by some major expressive examples.

Definition 3.1 Let W be a set of universe and $w_i \in W$ where $i \in I$. Let R be a NS over W. Then the subset of NS (sub-NS) R is denoted as $\varsigma_R(W^*)$ and defined as

$$\varsigma_R(W^*) = \left\{ \langle w_i, T_R(w_i), I_R(w_i), F_R(w_i) \rangle, \left\langle w_{i,j}, \max\left(T_R(w_i), T_R(w_j)\right), \max\left(I_R(w_i), I_R(w_j)\right), \min\left(F_R(w_i), F_R(w_j)\right) \right\rangle \right\}.$$

where $i, j \in I$ and $i \neq j$.

Clearly, $(w_{i,j}) = (w_{j,i})$.

Example 3.2 Let $W = \{w_1, w_2, w_3\}$ be a set of features of the refrigerator, where w_1 = energy efficiency, w_2 = capacity, w_3 =price. Let R be a NS over W, defined as

$$R = \{\langle w_1, .7, .5, .4 \rangle, \langle w_2, .2, .7, .9 \rangle, \langle w_3, .4, .1, .3 \rangle\}.$$

Then the sub-NS R is

$$\varsigma_R(W^*) = \{\langle w_1, .7, .5, .4 \rangle, \langle w_2, .2, .7, .9 \rangle, \langle w_3, .4, .1, .3 \rangle, \langle w_{1,2}, .7, .7, .4 \rangle, \langle w_{1,3}, .7, .5, .3 \rangle, \langle w_{2,3}, .4, .7, .3 \rangle\}.$$

Definition 3.3 Let W be a set of universe and $w_i \in W$ where $i \in I$. Let R be a NS over W. Then the subset of NS R with respect to w_i (sub-NS R_{w_i}) and w_i, w_j (sub-NS R_{w_i, w_j}) are denoted as $\varsigma_R(w_i)$ and $\varsigma_R(w_i, w_j)$, and defined as

$$\varsigma_R(w_i) = \left\{ \langle w_i, T_R(w_i), I_R(w_i), F_R(w_i) \rangle, \left\langle w_{i,j}, \max\left(T_R(w_i), T_R(w_j)\right), \max\left(I_R(w_i), I_R(w_j)\right), \min\left(F_R(w_i), F_R(w_j)\right) \right\rangle, \right.$$
$$\left. \langle w_k, T_R(0_n), I_R(0_n), F_R(0_n) \rangle, \langle w_{k,l}, T_R(0_n), I_R(0_n), F_R(0_n) \rangle \right\}$$

where $i \in I$, $j \in I - \{i\}$, $k, l \in I - \{i, j\}$ and $k \neq l$

and

$$\varsigma_R(w_i, w_j) = \left\{ \langle w_i, T_R(w_i), I_R(w_i), F_R(w_i) \rangle, \langle w_j, T_R(w_j), I_R(w_j), F_R(w_j) \rangle, \langle w_k, T_R(0_n), I_R(0_n), F_R(0_n) \rangle \right.$$
$$\left\langle w_{i,j}, \max\left(T_R(w_i), T_R(w_j)\right), \max\left(I_R(w_i), I_R(w_j)\right), \min\left(F_R(w_i), F_R(w_j)\right) \right\rangle,$$
$$\left\langle w_{i,k}, \max\left(T_R(w_i), T_R(w_k)\right), \max\left(I_R(w_i), I_R(w_k)\right), \min\left(F_R(w_i), F_R(w_k)\right) \right\rangle,$$
$$\left. \left\langle w_{j,k}, \max\left(T_R(w_j), T_R(w_k)\right), \max\left(I_R(w_j), I_R(w_k)\right), \min\left(F_R(w_j), F_R(w_k)\right) \right\rangle \right\}$$

where $i, j, k \in I$ and $i \neq j \neq k$, respectively.

Definition 3.4 Let W be a set of universe and $w \in W$. Let R be a NS over W and V be any proper non- empty subset of W. Then $\varsigma_R(V)$ is said to be neutro-fine set (NFS) over W.

Example 3.5 Consider Example 3.2.

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Then NFS $\varsigma_R(w_3)$ is defined as

$$\varsigma_R(w_3)=\{\langle w_3,.4,.1,.3\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.4,.7,.3\rangle\}.$$

That is,

$$\varsigma_R(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.4,.1,.3\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.4,.7,.3\rangle\}.$$

and NFS $\varsigma_R(w_1,w_2)$ is defined as

$$\varsigma_R(w_1,w_2)=\{\langle w_1,.7,.5,.4\rangle,\langle w_2,.2,.7,.9\rangle,\langle w_{1,2},.7,.7,.4\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.4,.7,.3\rangle\}.$$

That is,

$$\varsigma_R(w_1,w_2)=\{\langle w_1,.7,.5,.4\rangle,\langle w_2,.2,.7,.9\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.7,.7,.4\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.4,.7,.3\rangle\}.$$

Definition 3.6 Let *V* be any proper non- empty subset of *W*. Then the null NFS is denoted as 0_{nf} and defined as

$$0_{nf}=\{\langle V,T_R(V)=0,I_R(V)=0,F_R(V)=1\rangle:\forall V\}.$$

Definition 3.7 Let *V* be any proper non- empty subset of *W*. Then the absolute NFS is denoted as 1_{nf} and defined as

$$1_{nf}=\{\langle V,T_R(V)=1,I_R(V)=1,F_R(V)=0\rangle:\forall V\}.$$

Definition 3.8 Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then their union is denoted as $\varsigma_R(V_1)\cup\varsigma_R(V_2)=\varsigma_R(V_{1\vee 2})$ and is defined as

$$\varsigma_R(V_{1\vee 2})=\{\langle V_{1\vee 2},\max(T_R(V_1),T_R(V_2)),\max(I_R(V_1),I_R(V_2)),\min(F_R(V_1),F_R(V_2))\rangle:\forall V\subset W,V\neq\varphi\}.$$

Definition 3.9 Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then their intersection is denoted as $\varsigma_R(V_1)\cap\varsigma_R(V_2)=\varsigma_R(V_{1\wedge 2})$ and is defined as

$$\varsigma_R(V_{1\wedge 2})=\{\langle V_{1\wedge 2},\min(T_R(V_1),T_R(V_2)),\min(I_R(V_1),I_R(V_2)),\max(F_R(V_1),F_R(V_2))\rangle:\forall V\subset W,V\neq\varphi\}.$$

Definition 3.10 Let $\varsigma_R(V)$ be a NFS over *W*. Then its complement is denoted as $\varsigma_R(V)'$ and is defined as

$$\varsigma_R(V)'=\{\langle V,T_R(V),1-I_R(V),F_R(V)\rangle:\forall V\subset W,V\neq\varphi\}.$$

Clearly, $(\varsigma_R(V)')'=\varsigma_R(V)$.

Definition 3.11 Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then $\varsigma_R(V_1)$ is said to be a neutro-fine subset of $\varsigma_R(V_2)$ if

$$T_R(V_1)\le T_R(V_2),\ T_R(I_1)\le T_R(I_2),\ F_R(V_1)\ge F_R(V_2),\ \forall V\subset W,V\neq\varphi.$$

It is denoted by $\varsigma_R(V_1)\subseteq\varsigma_R(V_2)$.

Also $\varsigma_R(V_1)$ is said to be neutro-fine equal to $\varsigma_R(V_2)$ if $\varsigma_R(V_1)$ is a neutro-fine subset of $\varsigma_R(V_2)$ and $\varsigma_R(V_2)$ is a neutro-fine subset of $\varsigma_R(V_1)$. It is denoted by $\varsigma_R(V_1)=\varsigma_R(V_2)$.

Proposition 3.12 Let $\varsigma_R(V_1)$, $\varsigma_R(V_2)$ and $\varsigma_R(V_3)$ be NFSs over *W*. Then,

(i) $\varsigma_R(V_1)\cup 0_{nf}=\varsigma_R(V_1)$.

(ii) $\varsigma_R(V_1)\cup 1_{nf}=1_{nf}$.

(iii) $\varsigma_R(V_1)\cup[\varsigma_R(V_2)\cup\varsigma_R(V_3)]=[\varsigma_R(V_1)\cup\varsigma_R(V_2)]\cup\varsigma_R(V_3)$.

(iv) $\varsigma_R(V_1)\cup[\varsigma_R(V_2)\cap\varsigma_R(V_3)]=[\varsigma_R(V_1)\cup\varsigma_R(V_2)]\cap[\varsigma_R(V_1)\cup\varsigma_R(V_3)]$.

Proof. Straightforward.

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Proposition 3.13 Let $\varsigma_R(V_1)$, $\varsigma_R(V_2)$ and $\varsigma_R(V_3)$ be NFSs over *W*. Then,

(i) $\varsigma_R(V_1)\cap 0_{nf} = 0_{nf}$.

(ii) $\varsigma_R(V_1)\cap 1_{nf} = \varsigma_R(V_1)$.

(iii) $\varsigma_R(V_1)\cap[\varsigma_R(V_2)\cap\varsigma_R(V_3)] = [\varsigma_R(V_1)\cap\varsigma_R(V_2)]\cap\varsigma_R(V_3)$.

(iv) $\varsigma_R(V_1)\cap[\varsigma_R(V_2)\cup\varsigma_R(V_3)] = [\varsigma_R(V_1)\cap\varsigma_R(V_2)]\cup[\varsigma_R(V_1)\cap\varsigma_R(V_3)]$.

Proof. Straightforward.

Proposition 3.14 Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then,

(i) $[\varsigma_R(V_1)\cup\varsigma_R(V_2)]' = \varsigma_R(V_1)'\cap\varsigma_R(V_2)'$.

(ii) $[\varsigma_R(V_1)\cap\varsigma_R(V_2)]' = \varsigma_R(V_1)'\cup\varsigma_R(V_2)'$.

Proof. Straightforward.

Proposition 3.15 Let $\varsigma_R(V_1)$, $\varsigma_R(V_2)$ and $\varsigma_S(V_1)$ be NFSs over *W*. Then,

(i) $R\subseteq S \Rightarrow \varsigma_R(V_1)\subseteq\varsigma_S(V_1)$.

(ii) $\varsigma_R(V_1)\cup\varsigma_R(V_2) = \varsigma_R(V_1\cup V_2)$.

(iii) $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_1)$ and $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_2)$.

(iv) $\varsigma_R(V_1)\cup\varsigma_R(V_2)\supseteq\varsigma_R(V_1)$ and $\varsigma_R(V_1)\cup\varsigma_R(V_2)\supseteq\varsigma_R(V_2)$.

(v) $\varsigma_R(V_1)\subseteq\varsigma_R(V_2)\Rightarrow\varsigma_R(V_1)'\subseteq\varsigma_R(V_2)'$.

Proof. Straightforward.

Definition 3.16 Let *NFS*(*W*) be the family of all NFSs over *W*. Then the fine collection of $\varsigma_R(V)$ is denoted as ${}^f\varsigma_W$ and defined over the NT (W,τ_n) as

$${}^f\varsigma_W = \{0_{nf}, 1_{nf}, \cup\varsigma_R(V)\}.$$

Thus the triplet $(W,\tau_n,{}^f\varsigma_W)$ is said to be a neutro-fine topological space (NFTS) over (W,τ_n).

The elements belong to ${}^f\varsigma_W$ are said to be neutro-fine open sets (NFOSs) over (W,τ_n) and the complement of NFOSs are said to be neutro-fine closed sets (NFCSs) over (W,τ_n) and denote the collection by ${}^F\varsigma_W$.

Remark 3.17 If $(W,\tau_n,{}^f\varsigma_W)$ is a NFTS over (W,τ_n), then $(W,{}^f\varsigma_W)$ and $(W,{}^F\varsigma_W)$ are not NTSs over *W*.

Definition 3.18 Let *NFS*(*W*) be the family of all NFSs over *W*. Then ${}^f\varsigma_W$ is said be neutro-fine indiscrete topology if ${}^f\varsigma_W = \{0_{nf}, 1_{nf}\}$. Thus $(W,\tau_n,{}^f\varsigma_W)$ is said to be a neutro-fine indiscrete topological space over (W,τ_n).

Definition 3.19 Let *NFS*(*W*) be the family of all NFSs over *W*. Then ${}^f\varsigma_W$ is said be neutro-fine discrete topology if ${}^f\varsigma_W = NFS(W)$. Thus $(W,\tau_n,{}^f\varsigma_W)$ is said to be a neutro-fine discrete topological space over (W,τ_n).

Example 3.20 Let $W=\{w_1,w_2,w_3\}$ and $\tau_n=\{0_n,1_n,R,S\}$ where *R* and *S* are NSs over *W* and are defined as follows

$$R=\{\langle w_1,.1,.2,.8\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,.6,.5,.2\rangle\}$$

And

$$S=\{\langle w_1,.6,.5,.3\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,.7,.6,.1\rangle\}.$$

Thus (W,τ_n) is a NTS over *W*.

Then ${}^f\varsigma_W=\{0_n,1_n,\varsigma_R(w_1),\varsigma_R(w_2,w_3),\varsigma_S(w_2)\}$,

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where

$$\varsigma_R(w_1)=\{\langle w_1,.1,.2,.8\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.6,.5,.2\rangle,\langle w_{2,3},0,0,1\rangle\},$$

$$\varsigma_R(w_2,w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,.6,.5,.2\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.6,.5,.2\rangle,\langle w_{2,3},.6,.7,.2\rangle\},$$

$$\varsigma_S(w_2)=\{\langle w_1,0,0,1\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.9,.8,.1\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.9,.8,.1\rangle\}$$

are NFOSs over (W,τ_n).

Also, ${}^F\varsigma_W=\{0_n,1_n,\varsigma_R(w_1)',\varsigma_R(w_2,w_3)',\varsigma_S(w_2)'\}$,

where

$$\varsigma_R(w_1)'=\{\langle w_1,.8,.8,.1\rangle,\langle w_2,1,1,0\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.3,.3,.4\rangle,\langle w_{1,3},.2,.5,.6\rangle,\langle w_{2,3},1,1,0\rangle\},$$

$$\varsigma_R(w_2,w_3)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.3,.3,.4\rangle,\langle w_3,.2,.5,.6\rangle,\langle w_{1,2},.3,.3,.4\rangle,\langle w_{1,3},.2,.5,.6\rangle,\langle w_{2,3},.2,.3,.6\rangle\},$$

$$\varsigma_S(w_2)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.1,.2,.9\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.1,.2,.9\rangle,\langle w_{1,3},1,1,0\rangle,\langle w_{2,3},.1,.2,.9\rangle\}$$

are NFCSs over (W,τ_n).

Thus $(W,\tau_n,{}^f\varsigma_W)$ is a NFTS over (W,τ_n).

Since $\varsigma_R(w_1)\cup\varsigma_S(w_2)\notin{}^f\varsigma_W$ and $\varsigma_R(w_1)\cap\varsigma_S(w_2)\notin{}^f\varsigma_W$, ${}^f\varsigma_W$ does not satisfy the conditions of a NT.

Also, since $\varsigma_R(w_1)'\cup\varsigma_S(w_2)'\notin{}^F\varsigma_W$ and $\varsigma_R(w_1)'\cap\varsigma_S(w_2)'\notin{}^F\varsigma_W$, ${}^F\varsigma_W$ does not satisfy the conditions of a NT.

Hence $(W,{}^f\varsigma_W)$ and $(W,{}^F\varsigma_W)$ are not NTSs over W.

Definition 3.21 Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine interior of $\varsigma_R(V)$ is denoted as $Int_{nf}(\varsigma_R(V))$ and is defined as the union of all NFOSs contained in $\varsigma_R(V)$.

Clearly, $Int_{nf}(\varsigma_R(V))$ is the largest NFOS contained in $\varsigma_R(V)$.

Definition 3.22 Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine closure of $\varsigma_R(V)$ is denoted as $Cl_{nf}(\varsigma_R(V))$ and is defined as the intersection of all NFCSs containing $\varsigma_R(V)$.

Clearly, $Cl_{nf}(\varsigma_R(V))$ is the smallest NFCS containing $\varsigma_R(V)$.

Example 3.23 Consider Example 3.20.

Let us consider the NS R.

Then the NFS $\varsigma_R(w_1,w_3)$ is defined as

$$\varsigma_R(w_1,w_3)=\{\langle w_1,.1,.2,.8\rangle,\langle w_2,.0,0,1\rangle,\langle w_3,.6,.5,.2\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.6,.5,.2\rangle,\langle w_{2,3},.6,.7,.2\rangle\}.$$

Clearly,

$$\varsigma_R(w_1,w_3)\supseteq 0_n,\varsigma_R(w_1)\ .$$

Thus

$$Int_{nf}(\varsigma_R(w_1,w_3))=0_n\cup\varsigma_R(w_1)=\varsigma_R(w_1)\ .$$

The NFS $\varsigma_R(w_2)$ is defined as

$$\varsigma_R(w_2)=\{\langle w_1,.0,0,1\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,.0,0,1\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.0,0,1\rangle,\langle w_{2,3},.6,.7,.2\rangle\}.$$

Clearly,

$$\varsigma_R(w_2)\subseteq 1_n,\varsigma_R(w_1)'\ .$$

Thus

$$Cl_{nf}(\varsigma_R(w_2))=1_n\cap\varsigma_R(w_1)'=\varsigma_R(w_1)'\ .$$

Now consider the NS S.

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Then the NFS $\varsigma_S(w_2,w_3)$ is defined as

$$\varsigma_S(w_2,w_3)=\left\{\langle w_1,0,0,1\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,.7,.6,.5\rangle,\langle w_{1,2},.9,.8,.1\rangle,\langle w_{1,3},.7,.6,.1\rangle,\langle w_{2,3},.9,.8,.1\rangle\right\}.$$

Clearly,

$$\varsigma_S(w_2,w_3)\supseteq 0_n,\varsigma_S(w_2) \ .$$

Thus

$$Int_{nf}(\varsigma_S(w_2,w_3))=0_n\cup\varsigma_S(w_2)=\varsigma_S(w_2) \ .$$

The NFS $\varsigma_S(w_1)$ is defined as

$$\varsigma_S(w_1)=\left\{\langle w_1,.6,.5,.3\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.9,.8,.1\rangle,\langle w_{1,3},.7,.6,.1\rangle,\langle w_{2,3},0,0,1\rangle\right\}.$$

Clearly,

$$\varsigma_S(w_1)\subseteq 1_n,\varsigma_S(w_2)' \ .$$

Thus

$$Cl_{nf}(\varsigma_S(w_1))=1_n\cap\varsigma_S(w_2)'=\varsigma_S(w_2)' \ .$$

Proposition 3.24 Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then,

(i) $Int_{nf}(0_{nf})=0_{nf}$ and $Int_{nf}(1_{nf})=1_{nf}$.

(ii) $\varsigma_R(V_1)$ is NFOS $\Rightarrow Int_{nf}(\varsigma_R(V_1))=\varsigma_R(V_1)$.

(iii) $Int_{nf}(\varsigma_R(V_1))\subseteq\varsigma_R(V_1)$.

(iv) $\varsigma_R(V_1)\subseteq\varsigma_R(V_2)\Rightarrow Int_{nf}(\varsigma_R(V_1))\subseteq Int_{nf}(\varsigma_R(V_2))$.

(v) $Int_{nf}(Int_{nf}(\varsigma_R(V_1)))=Int_{nf}(\varsigma_R(V_1))$.

(vi) $Int_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))=Int_{nf}(\varsigma_R(V_1))\cap Int_{nf}(\varsigma_R(V_2))$.

(vii) $Int_{nf}(\varsigma_R(V_1)\cup\varsigma_R(V_2))\subseteq Int_{nf}(\varsigma_R(V_1))\cup Int_{nf}(\varsigma_R(V_2))$.

(viii) $Int_{nf}(\varsigma_R(V_1)')=\left[Cl_{nf}(\varsigma_R(V_1))\right]'$.

Proof. Straightforward.

Proposition 3.25 Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then,

(i) $Cl_{nf}(0_{nf})=0_{nf}$ and $Cl_{nf}(1_{nf})=1_{nf}$.

(ii) $\varsigma_R(V_1)$ is NFCS $\Rightarrow Cl_{nf}(\varsigma_R(V_1))=\varsigma_R(V_1)$.

(iii) $Cl_{nf}(\varsigma_R(V_1))\supseteq\varsigma_R(V_1)$.

(iv) $\varsigma_R(V_1)\subseteq\varsigma_R(V_2)\Rightarrow Cl_{nf}(\varsigma_R(V_1))\subseteq Cl_{nf}(\varsigma_R(V_2))$.

(v) $Cl_{nf}(Cl_{nf}(\varsigma_R(V_1)))=Cl_{nf}(\varsigma_R(V_1))$

(vi) $Cl_{nf}(\varsigma_R(V_1)\cup\varsigma_R(V_2))=Cl_{nf}(\varsigma_R(V_1))\cup Cl_{nf}(\varsigma_R(V_2))$.

(vii) $Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq Cl_{nf}(\varsigma_R(V_1))\cap Cl_{nf}(\varsigma_R(V_2))$.

(viii) $Cl_{nf}(\varsigma_R(V_1)')=\left[Int_{nf}(\varsigma_R(V_1))\right]'$.

Proof. Straightforward.

4. Separation Axioms

In this section, separation axioms on NFTS are defined with examples.

Definition 4.1 Let *NF*(*W*) be the family of all NFs over the universe *W* and $w\in W$. Then NFS $w^{\langle\alpha,\beta,\chi\rangle}$ is said to be a neutro-fine point (NFP), for $0\le\alpha,\beta,\gamma\le 1$ and is defined as follows:

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$$w^{\langle\alpha,\beta,\gamma\rangle}(v)=\begin{cases}(\alpha,\beta,\gamma), & if \quad w=v\\ (0,0,1), & if \quad w\neq v\end{cases}.$$

Every NFS is the union of its NFPs.

Definition 4.2 Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W, τ_η). Let $\varsigma_R(V)$ be a NFS over W. Then $w^{\langle\alpha,\beta,\chi\rangle}$ belongs to the NFS $\varsigma_R(V)$ is denoted as $w^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V)$ and is defined as $\alpha\le T_R(V)$, $\beta\le I_R(V)$, and $\gamma\ge F_R(V)$.

Example 4.3 Let $W=\{w_1,w_2,w_3\}$. Let *R* be a NS over *W*, defined as

$$R=\left\{\langle w_1,.7,.5,.4\rangle,\langle w_2,.2,.7,.9\rangle,\langle w_3,.4,.1,.3\rangle\right\}.$$

Then the NFS $\varsigma_R(w_{1,3})$ is defined as

$$\varsigma_R(w_{1,3})=\left\{w_{1,3}{}^{\langle .7,.5,.3\rangle}\right\}$$
$$=\left\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},0,0,1\rangle\right\}.$$

Thus the NFS $\varsigma_R(w_{1,3})$ is a NFP.

Definition 4.4 Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over *W*. Then $\varsigma_R(V)$ is said to be a neutro-fine neighborhood of the NFP $w^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V)$, if there exists a NFOS $\varsigma_R(U)$ such that $w^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(U)\subseteq\varsigma_R(V)$.

Definition 4.5 Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ be two NFPs over W. Then $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ are said to be distinct points if $u^{\langle\alpha,\beta,\chi\rangle}\cap v^{\langle\alpha,\beta,\chi\rangle}=0_{nf}$.

Definition 4.6 Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ be any distinct NFPs. If there exists NFOSs $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ such that

$$u^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V_1) \quad \text{and} \quad u^{\langle\alpha,\beta,\chi\rangle}\cap\varsigma_R(V_2)=0_{nf} \quad \text{or}$$
$$v^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V_2) \quad \text{and} \quad v^{\langle\alpha,\beta,\chi\rangle}\cap\varsigma_R(V_1)=0_{nf}.$$

Then $\left(W,\tau_n,{}^f\varsigma_W\right)$ is said to be a neutro-fine T_0-space.

Definition 4.7 Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ be any distinct NFPs. If there exists NFOSs $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ such that

$$u^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V_1) \quad \text{and} \quad u^{\langle\alpha,\beta,\chi\rangle}\cap\varsigma_R(V_2)=0_{nf} \quad \text{and}$$
$$v^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V_2) \quad \text{and} \quad v^{\langle\alpha,\beta,\chi\rangle}\cap\varsigma_R(V_1)=0_{nf}.$$

Then $\left(W,\tau_n,{}^f\varsigma_W\right)$ is said to be a neutro-fine T_1-space.

Theorem 4.8 Every neutro-fine T_1-space is neutro-fine T_0-space.
The proof follows from Definitions 4.6 and 4.7.

Remark 4.9 The converse of the above theorem is not true as shown in the following example.

Example 4.10 Let $W=\{w_1,w_2,w_3\}$ and $\tau_n=\{0_n,1_n,R,S\}$ where *R* and *S* are NSs over *W* and are defined as follows

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$$R = \{\langle w_1,.5,.3,.2\rangle, \langle w_2,.9,.6,.1\rangle, \langle w_3,.6,.5,.4\rangle\}$$

And

$$S = \{\langle w_1,.3,.1,.5\rangle, \langle w_2,.7,.3,.4\rangle, \langle w_3,.2,.3,.8\rangle\}.$$

Thus (W, τ_n) is a NTS over W.

Consider $w_2^{\langle .9,.6,.1\rangle}$, $w_{2,3}^{\langle .2,.3,.4\rangle}$ and $w_3^{\langle .2,.3,.8\rangle}$ are NFPs.

Then ${}^f\varsigma_W = \{0_n, 1_n, \varsigma_R(w_2), \varsigma_S(w_2, w_3), \varsigma_S(w_3)\}$, where

$$\varsigma_R(w_2) = \{w_2^{\langle .9,.6,.1\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,.9,.6,.1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},0,0,1\rangle\},$$

$$\varsigma_S(w_{2,3}) = \{w_2^{\langle .9,.6,.1\rangle} \cup w_{2,3}^{\langle .2,.3,.4\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,.9,.6,.1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},.2,.3,.4\rangle\},$$

$$\varsigma_S(w_3) = \{w_3^{\langle .2,.3,.8\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,.2,.3,.8\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},0,0,1\rangle\}.$$

Thus $(W, \tau_n, {}^f\varsigma_W)$ is a NFTS over (W, τ_n) and also a neutro-fine T_0 -space.

Here

$$w_2^{\langle .9,.6,.1\rangle} \in \varsigma_R(w_2),\ w_2^{\langle .9,.6,.1\rangle} \cap \varsigma_S(w_{2,3}) \neq 0_{nf} \text{ and}$$
$$w_{2,3}^{\langle .2,.3,.4\rangle} \in \varsigma_s(w_{2,3}),\ w_{2,3}^{\langle .2,.3,.4\rangle} \cap \varsigma_R(w_2) \neq 0_{nf}.$$

Thus $(W, \tau_n, {}^f\varsigma_W)$ is not a neutro-fine T_1 -space because for NFPs $w_2^{\langle .9,.6,.1\rangle}$ and $w_{2,3}^{\langle .2,.3,.4\rangle}$.

Hence $(W, \tau_n, {}^f\varsigma_W)$ is a neutro-fine T_0 -space but not a neutro-fine T_1 -space.

Definition 4.11 Let $(W, \tau_n, {}^f\varsigma_W)$ be a NFTS over (W, τ_n). Let $u^{\langle \alpha,\beta,\chi\rangle}$ and $v^{\langle \alpha,\beta,\chi\rangle}$ be any distinct NFPs. If there exists NFOSs $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ such that

$$u^{\langle \alpha,\beta,\chi\rangle} \in \varsigma_R(V_1),\quad v^{\langle \alpha,\beta,\chi\rangle} \in \varsigma_R(V_2) \text{ and } \quad \varsigma_R(V_1) \cap \varsigma_R(V_2) = 0_{nf}.$$

Then $(W, \tau_n, {}^f\varsigma_W)$ is said to be a neutro-fine T_2 -space.

Theorem 4.12 Every neutro-fine T_2 -space is neutro-fine T_1 -space.
The proof follows from Definitions 4.7 and 4.11.

Example 4.13 Consider Example 4.10.

Consider $w_3^{\langle .6,.5,.4\rangle}$, $w_{1,2}^{\langle .9,.6,.1\rangle}$, $w_{2,3}^{\langle .2,.3,.4\rangle}$ and $w_{1,3}^{\langle .3,.3,.5\rangle}$ are NFPs.

Then ${}^f\varsigma_W = \{0_n, 1_n, \varsigma_R(w_3), \varsigma_R(w_1, w_2), \varsigma_S(w_2, w_3), \varsigma_S(w_1, w_3)\}$, where

$$\varsigma_R(w_3) = \{w_3^{\langle .6,.5,.4\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,.6,.5,.4\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},0,0,1\rangle\},$$

$$\varsigma_R(w_{1,2}) = \{w_{1,2}^{\langle .9,.6,.1\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},.9,.6,.1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},0,0,1\rangle\},$$

$$\varsigma_S(w_{2,3}) = \{w_{2,3}^{\langle .2,.3,.4\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},.2,.3,.4\rangle\},$$

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$$\varsigma_S(w_{1,3}) = \left\{w_{1,3}{}^{\langle .3,.3,.5\rangle}\right\}$$
$$= \left\{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},.3,.3,.5\rangle, \langle w_{2,3},0,0,1\rangle\right\}.$$

Thus $\left(W, \tau_n, {}^f\varsigma_W\right)$ is a NFTS over (W, τ_n).
Hence $\left(W, \tau_n, {}^f\varsigma_W\right)$ is a neutro-fine T_2-space and neutro-fine T_1-space as well as neutro-fine T_0-space.

Theorem 4.14 Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS over (W, τ_n). Then $\left(W, \tau_n, {}^f\varsigma_W\right)$ is a neutro-fine T_1-space if and only if each NFP is a NFCS.
Proof. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a neutro-fine T_1-space and $u^{\langle\alpha,\beta,\chi\rangle}$ be any NFP.
To prove: $\left(u^{\langle\alpha,\beta,\chi\rangle}\right)'$ is a NFCS.
Let $v^{\langle\alpha,\beta,\chi\rangle} \in \left(u^{\langle\alpha,\beta,\chi\rangle}\right)'$.
Then $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ are distinct NFPs.
Thus $u^{\langle\alpha,\beta,\chi\rangle} \cap v^{\langle\alpha,\beta,\chi\rangle} = 0_{nf}$.
Since $\left(W, \tau_n, {}^f\varsigma_W\right)$ is a neutro-fine T_1-space, there exists a NFOS $\varsigma_R(V)$ such that

$v^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V)$ and $v^{\langle\alpha,\beta,\chi\rangle} \cap \varsigma_R(V_1) = 0_{nf}$.

Then $v^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V) \subseteq \left(u^{\langle\alpha,\beta,\chi\rangle}\right)'$.
Thus $\left(u^{\langle\alpha,\beta,\chi\rangle}\right)'$ is a NFOS.
Hence $u^{\langle\alpha,\beta,\chi\rangle}$ is a NFCS.
Conversely, suppose that each NFP $u^{\langle\alpha,\beta,\chi\rangle}$ is a NFCS.
Then $\left(u^{\langle\alpha,\beta,\chi\rangle}\right)'$ is a NFOS.
Let $u^{\langle\alpha,\beta,\chi\rangle} \cap v^{\langle\alpha,\beta,\chi\rangle} = 0_{nf}$.
Thus $v^{\langle\alpha,\beta,\chi\rangle} \in \left(u^{\langle\alpha,\beta,\chi\rangle}\right)'$ and $v^{\langle\alpha,\beta,\chi\rangle} \cap \left(u^{\langle\alpha,\beta,\chi\rangle}\right)' = 0_{nf}$.
Hence $\left(W, \tau_n, {}^f\varsigma_W\right)$ is a neutro-fine T_1-space over (W, τ_n).

Theorem 4.15 Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS over (W, τ_n). $\left(W, \tau_n, {}^f\varsigma_W\right)$ is a neutro-fine T_2-space if and only if for distinct NFPs $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$, there exists a NFOS $\varsigma_R(V)$ containing $u^{\langle\alpha,\beta,\chi\rangle}$ but not $v^{\langle\alpha,\beta,\chi\rangle}$ such that $v^{\langle\alpha,\beta,\chi\rangle} \notin Cl_{nf}(\varsigma_R(V))$.
Proof. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a neutro-fine T_2-space.
Let $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ be two NFPs.
Then there exists disjoint NFOSs $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ such that

$u^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V_1)$ and $v^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V_2)$.

Since $u^{\langle\alpha,\beta,\chi\rangle} \cap v^{\langle\alpha,\beta,\chi\rangle} = 0_{nf}$ and $\varsigma_R(V_1) \cap \varsigma_R(V_2) = 0_{nf}$, then

$v^{\langle\alpha,\beta,\chi\rangle} \notin \varsigma_R(V_1)$.

Hence $v^{\langle\alpha,\beta,\chi\rangle} \notin Cl_{nf}(\varsigma_R(V_1))$.

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Conversely, suppose that for distinct NFPs $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$, there exists a NFOS $\varsigma_R(V)$ containing $u^{\langle\alpha,\beta,\chi\rangle}$ but not $v^{\langle\alpha,\beta,\chi\rangle}$ such that $v^{\langle\alpha,\beta,\chi\rangle} \notin Cl_{nf}(\varsigma_R(V))$.

Then $v^{\langle\alpha,\beta,\chi\rangle} \in (Cl_{nf}(\varsigma_R(V)))'$.

Thus $\varsigma_R(V)$ and $(Cl_{nf}(\varsigma_R(V)))'$ are disjoint NFOSs containing $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ respectively.

Hence $(W,\tau_n,{}^f\varsigma_W)$ is a neutro-fine T_2-space.

Theorem 4.16 Let $(W,\tau_n,{}^f\varsigma_W)$ be a neutro-fine T_1-space for every distinct NFPs $u^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V_1) \in {}^f\varsigma_W$. If there exists a NFOS $\varsigma_R(V_2)$ such that

$$u^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V_2) \subseteq Cl_{nf}(\varsigma_R(V_2)) \subseteq \varsigma_R(V_1),$$

then $(W,\tau_n,{}^f\varsigma_W)$ is a neutro-fine T_2-space.

Proof. Let $(W,\tau_n,{}^f\varsigma_W)$ be a neutro-fine T_1-space.

Suppose that $u^{\langle\alpha,\beta,\chi\rangle} \cap v^{\langle\alpha,\beta,\chi\rangle} = 0_{nf}$.

Since $(W,\tau_n,{}^f\varsigma_W)$ is a neutro-fine T_1-space, $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ are NFCSs in ${}^f\varsigma_W$.

Thus $u^{\langle\alpha,\beta,\chi\rangle} \in \left(v^{\langle\alpha,\beta,\chi\rangle}\right)' \in {}^f\varsigma_W$.

Then there exists a NFOS $\varsigma_R(V_2)$ such that

$$u^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V_2) \subseteq Cl_{nf}(\varsigma_R(V_2)) \subseteq \left(v^{\langle\alpha,\beta,\chi\rangle}\right)'.$$

Thus

$$v^{\langle\alpha,\beta,\chi\rangle} \in (Cl_{nf}(\varsigma_R(V_2)))',\ u^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V_2) \text{ and } \varsigma_R(V_2) \cap (Cl_{nf}(\varsigma_R(V_2)))' = 0_{nf}.$$

Hence $(W,\tau_n,{}^f\varsigma_W)$ is a neutro-fine T_2-space.

5. Decision Making in NFTS

In this section, the real-life application is intimated to take a correct decision on DM problems and an example is investigated in two different manners.

Definition 5.1 Let $\varsigma_R(V)$ be a NFS over W of a NFTS $(W,\tau_n,{}^f\varsigma_W)$. Then the absolute complement of $\varsigma_R(V)$ is denoted as $\varsigma_R{}^*(V')$ and defined as $\varsigma_R{}^*(V') = \varsigma_R{}^*(W-V)$.

Thus the collection of $\varsigma_R{}^*(V')$ is denoted as ${}^{f*}\varsigma_W$ and defined as ${}^{f*}\varsigma_W = \{0_{nf}, 1_{nf}, \bigcup \varsigma_R{}^*(V')\}$. The elements belong to ${}^{f*}\varsigma_W$ are said to be neutro-fine absolute open sets (NFAOSs) over (W,τ_n) and the complement of NFOSs are said to be neutro-fine absolute closed sets (NFACSs) over (W,τ_n) and denote the collection by ${}^{F*}\varsigma_W$.

Example 5.2 Let $W=\{w_1,w_2,w_3\}$ and $\tau_n=\{0_n,1_n,R\}$ where R is a NS over W and are defined as follows

$$R=\{\langle w_1,.9,.4,.6\rangle,\langle w_2,.6,.5,.1\rangle,\langle w_3,.7,.8,.1\rangle\}.$$

Thus (W,τ_n) is a NTS over W.

Then NFOSs over (W,τ_n) are ${}^f\varsigma_W=\{0_n,1_n,\varsigma_R(w_3),\varsigma_R(w_2,w_3)\}$, where

$$\varsigma_R(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.7,.8,.1\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.9,.8,.1\rangle,\langle w_{2,3},.7,.8,.1\rangle\} \text{ and}$$

$$\varsigma_R(w_2,w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,.6,.5,.1\rangle,\langle w_3,.7,.8,.1\rangle,\langle w_{1,2},.9,.5,.1\rangle,\langle w_{1,3},.9,.8,.1\rangle,\langle w_{2,3},.7,.8,.1\rangle\}.$$

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Also, NFCSs over (W,τ_n) are ${}^{F}\varsigma_W = \{0_n, 1_n, \varsigma_R(w_3)' \varsigma_R(w_2,w_3)'\}$, where

$$\varsigma_R(w_3)' = \{\langle w_1,1,1,0\rangle, \langle w_2,1,1,0\rangle, \langle w_3,.1,.2,.7\rangle, \langle w_{1,2},1,1,0\rangle, \langle w_{1,3},.1,.2,.9\rangle, \langle w_{2,3},.1,.2,.7\rangle\} \text{ and}$$

$$\varsigma_R(w_2,w_3)' = \{\langle w_1,1,1,0\rangle, \langle w_2,.1,.5,.6\rangle, \langle w_3,.1,.2,.7\rangle, \langle w_{1,2},.1,.5,.9\rangle, \langle w_{1,3},.1,.2,.9\rangle, \langle w_{2,3},.1,.2,.7\rangle\}.$$

Thus $(W,\tau_n, {}^{f}\varsigma_W)$ is a NFTS over (W,τ_n).

Then NFAOSs over (W,τ_n) are

$${}^{f*}\varsigma_W = \{0_n, 1_n, \varsigma_R{}^*((w_3)'), \varsigma_R{}^*((w_2,w_3)')\} = \{0_n, 1_n, \varsigma_R{}^*(w_1,w_2), \varsigma_R{}^*(w_1)\},$$

where

$$\varsigma_R{}^*((w_3)') = \varsigma_R{}^*(w_1,w_2) = \{\langle w_1,.9,.4,.6\rangle, \langle w_2,.6,.5,.1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},.9,.4,.1\rangle, \langle w_{1,3},.9,.8,.1\rangle, \langle w_{2,3},.7,.8,.1\rangle\} \text{ and}$$

$$\varsigma_R{}^*((w_2,w_3)') = \varsigma_R{}^*(w_1) = \{\langle w_1,.9,.4,.6\rangle, \langle w_2,0,0,1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},.9,.5,.1\rangle, \langle w_{1,3},.9,.8,.1\rangle, \langle w_{2,3},0,0,1\rangle\}.$$

Also, NFACSs over (W,τ_n) are

$${}^{F*}\varsigma_W = \{0_n, 1_n, \varsigma_R{}^*((w_3)')', \varsigma_R{}^*((w_2,w_3)')'\} = \{0_n, 1_n, \varsigma_R{}^*(w_1,w_2)', \varsigma_R{}^*(w_1)'\},$$

where

$$\varsigma_R{}^*((w_3)')' = \varsigma_R{}^*(w_1,w_2)' = \{\langle w_1,.6,.6,.9\rangle, \langle w_2,.1,.5,.6\rangle, \langle w_3,1,1,0\rangle, \langle w_{1,2},.1,.6,.9\rangle, \langle w_{1,3},.1,.2,.9\rangle, \langle w_{2,3},.1,.2,.7\rangle\}$$

and

$$\varsigma_R{}^*((w_2,w_3)')' = \varsigma_R{}^*(w_1)' = \{\langle w_1,.6,.6,.9\rangle, \langle w_2,1,1,0\rangle, \langle w_3,1,1,0\rangle, \langle w_{1,2},.1,.5,.9\rangle, \langle w_{1,3},.1,.2,.9\rangle, \langle w_{2,3},1,1,0\rangle\}.$$

Definition 5.3 Let W be a set of universe and $w \in W$. Let R be a NS over W and V be any proper non-empty subset of W. Let $\varsigma_R(V)$ be a NFS over W of a NFTS $(W,\tau_n, {}^{f}\varsigma_W)$. Then the net value of R is calculated by the formula

Then the net value of R (NV(R)) is calculated by the formula

$$NV(R) = \left| \frac{\left[\sum_i (T_R{}^*(V'))_i - \sum_i (F_R(V)')_i\right] + \left[\sum_i (T_R(V)')_i - \sum_i (F_R{}^*(V'))_i\right]}{2} \times \left[1 - \frac{\sum_i (I_R{}^*(V'))_i - \sum_i (I_R(V)')_i}{2}\right] \right| \tag{5.1}$$

where

$\sum_i (T_R(V)')_i$, $\sum_i (I_R(V)')_i$ and $\sum_i (F_R(V)')_i$ are the sum of all truth, indeterminacy and falsity values of $\varsigma_R(V)'$ respectively, and

$\sum_i (T_R{}^*(V'))_i$, $\sum_i (I_R{}^*(V'))_i$ and $\sum_i (F_R{}^*(V'))_i$ are the sum of all truth, indeterminacy, and falsity values of $\varsigma_R{}^*(V')$ respectively.

Algorithm

Step 1: List the set of items for the sample.

Step 2: List some of its risk factors as the universe W, where $w \in W$.

Step 3: Go through the damages of the items.

Step 4: Define each item as NSs, say R.

Step 5: Collect these NSs which defines a NT τ_n and so (W,τ_n) is a NTS.

Step 6: List the proper subsets of W as V.

Step 7: Define NFSs for each NS with respect to their risk factors $v \in V$, say $\varsigma_R(V)$.

Step 8: Define a neutrosophic-fine collection ${}^{f}\varsigma_W$, where $\varsigma_R(v)$ are NFOSs and so $(W,\tau_n, {}^{f}\varsigma_W)$ is a NFTS.

Step 9: Find the complement and absolute complement of ${}^{f}\varsigma_W$, ${}^{F}\varsigma_W$ and ${}^{f*}\varsigma_W$ respectively, which represents the secureness of the items.

Step 10: Calculate the NV(R) by using the formula (5.1).

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Step 11: Select the highest value of NV(R) among all the calculated values of NV(R).
Step 12: If two or more NV(R)s are identical for a particular w, replace that w with some other risk factor and recurrence the procedure.
Step 13: Terminate the procedure, while attaining a unique NV(R).

Example 5.4 Consider the problem that a customer wishes to buy a second-hand refrigerator. Let $R1$, $R2$, $R3$, and $R4$ be sample refrigerators for second-hand sales. Each refrigerator is damaged according to some aspects of the universe $W=\{w_1,w_2,w_3\}$, where w_1 –locked compressor, w_2 –clogged coils and w_3 –dirty condenser coil. Our problem is to help the customer to prefer a second-hand refrigerator with less damage.

1. Let $R1$, $R2$, $R3$, and $R4$ be sample refrigerators for second-hand sales.
2. Let $W=\{w_1,w_2,w_3\}$ be the universe, where w_1 –locked compressor, w_2 –clogged coils and w_3 –dirty condenser coil.
3. Analyze the damages on each refrigerator.
4. Define $R1$, $R2$, $R3$, and $R4$ as NSs.

$$R1=\{\langle w_1,.6,.4,.7\rangle,\langle w_2,.5,.6,.7\rangle,\langle w_3,.3,.4,.1\rangle\},$$
$$R2=\{\langle w_1,.3,.5,.8\rangle,\langle w_2,.4,.8,.5\rangle,\langle w_3,.2,.3,.7\rangle\},$$
$$R3=\{\langle w_1,.6,.5,.7\rangle,\langle w_2,.5,.8,.5\rangle,\langle w_3,.3,.4,.1\rangle\} \text{ and}$$
$$R4=\{\langle w_1,.3,.4,.8\rangle,\langle w_2,.4,.6,.7\rangle,\langle w_3,.2,.3,.7\rangle\}.$$

5. Thus $\tau_n=\{0_n,1_n,R1,R2,R3,R4\}$ is a NT and so (W,τ_n) is a NTS.
6. Let $V=\{(w_1),(w_2),(w_3),(w_1,w_2),(w_1,w_3),(w_2,w_3)\}$ be the set of proper subsets of W.
7. Define NFSs as

$$\varsigma_{R1}(w_1,w_2)=\{\langle w_1,.6,.4,.7\rangle,\langle w_2,.5,.6,.7\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.6,.6,.7\rangle,\langle w_{1,3},.6,.4,.1\rangle,\langle w_{2,3},.5,.6,.1\rangle\},$$
$$\varsigma_{R2}(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.2,.3,.7\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.3,.5,.7\rangle,\langle w_{2,3},.4,.8,.5\rangle\},$$
$$\varsigma_{R3}(w_2)=\{\langle w_1,0,0,1\rangle,\langle w_2,.5,.8,.5\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.6,.8,.5\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.5,.8,.1\rangle\} \text{ and}$$
$$\varsigma_{R4}(w_1,w_3)=\{\langle w_1,.3,.4,.8\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.2,.3,.7\rangle,\langle w_{1,2},.4,.6,.7\rangle,\langle w_{1,3},.3,.4,.7\rangle,\langle w_{2,3},.4,.6,.7\rangle\}.$$

8. Then the neutro-fine collection ${}^f\varsigma_W=\{0_n,1_n,\varsigma_{R1}(w_1,w_2),\varsigma_{R2}(w_3),\varsigma_{R3}(w_2),\varsigma_{R4}(w_1,w_3)\}$, whose elements are NFOSs.

Thus $(W,\tau_n,{}^f\varsigma_W)$ is a NFTS over (W,τ_n).

9. Let the complement and absolute complement of ${}^f\varsigma_W$ represents the secureness of the items.

The complement of ${}^f\varsigma_W$ is ${}^F\varsigma_W=\{0_n,1_n,\varsigma_{R1}(w_1,w_2)',\varsigma_{R2}(w_3)',\varsigma_{R3}(w_2)',\varsigma_{R4}(w_1,w_3)'\}$, where

$$\varsigma_{R1}(w_1,w_2)'=\{\langle w_1,.7,.6,.6\rangle,\langle w_2,.7,.4,.5\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.7,.4,.6\rangle,\langle w_{1,3},.1,.6,.6\rangle,\langle w_{2,3},.1,.4,.5\rangle\},$$
$$\varsigma_{R2}(w_3)'=\{\langle w_1,1,1,0\rangle,\langle w_2,1,1,0\rangle,\langle w_3,.7,.7,.2\rangle,\langle w_{1,2},1,1,0\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.5,.2,.4\rangle\},$$
$$\varsigma_{R3}(w_2)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.5,.2,.5\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.5,.2,.6\rangle,\langle w_{1,3},1,1,0\rangle,\langle w_{2,3},.1,.2,.5\rangle\} \text{ and}$$
$$\varsigma_{R4}(w_1,w_3)'=\{\langle w_1,.8,.6,.3\rangle,\langle w_2,1,1,0\rangle,\langle w_3,.7,.7,.2\rangle,\langle w_{1,2},.7,.4,.4\rangle,\langle w_{1,3},.7,.6,.3\rangle,\langle w_{2,3},.7,.4,.4\rangle\}.$$

The absolute complement of ${}^f\varsigma_W$ is ${}^{f*}\varsigma_W=\{0_n,1_n,\varsigma_{R1}{}^*(w_3),\varsigma_{R2}{}^*(w_1,w_2),\varsigma_{R3}{}^*(w_1,w_3),\varsigma_{R4}{}^*(w_2)\}$, where

$$\varsigma_{R1}{}^*(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.3,.4,.1\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.6,.4,.1\rangle,\langle w_{2,3},.5,.6,.1\rangle\},$$
$$\varsigma_{R2}{}^*(w_1,w_2)=\{\langle w_1,.3,.5,.8\rangle,\langle w_2,.4,.8,.5\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.4,.8,.5\rangle,\langle w_{1,3},.3,.5,.7\rangle,\langle w_{2,3},.4,.8,.5\rangle\},$$
$$\varsigma_{R3}{}^*(w_1,w_3)=\{\langle w_1,.6,.5,.7\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.3,.4,.1\rangle,\langle w_{1,2},.6,.8,.5\rangle,\langle w_{1,3},.6,.5,.1\rangle,\langle w_{2,3},.5,.8,.1\rangle\} \text{ and}$$
$$\varsigma_{R4}{}^*(w_2)=\{\langle w_1,0,0,1\rangle,\langle w_2,.4,.6,.7\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.4,.6,.7\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.4,.6,.7\rangle\}.$$

10. By using the formula (5.1), the following values are obtained.

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NV(R1) = 0.98,
NV(R2) = 2.61,
NV(R3) = **2.99** and
NV(R4) = 0.79.

11. Thus NV(R3) is the highest value.

Hence the customer can prefer to buy R3 for the second use.

Example 5.5 Consider the situation of Example 5.4.

1. Let $R1$, $R2$, and $R3$ be sample refrigerators for second-hand sale.
2. Let $W = \{w_1, w_2, w_3\}$ be the universe, where w_1 –locked compressor, w_2 –clogged coils and w_3 –dirty condenser coil.
3. Analyze the damages on each refrigerator.
4. Define $R1$, $R2$, and $R3$ as NSs.

$R1 = \{\langle w_1, .3, .4, .5\rangle, \langle w_2, .5, .6, .3\rangle, \langle w_3, .1, .2, .3\rangle\}$,
$R2 = \{\langle w_1, .4, .5, .2\rangle, \langle w_2, .6, .8, .3\rangle, \langle w_3, .4, .5, .2\rangle\}$ and
$R3 = \{\langle w_1, .4, .5, .2\rangle, \langle w_2, .6, .8, .3\rangle, \langle w_3, .4, .5, .2\rangle\}$.

5. Thus $\tau_n = \{0_n, 1_n, R1, R2, R3\}$ is a NT and so (W, τ_n) is a NTS.
6. Let $V = \{(w_1), (w_2), (w_3), (w_1, w_2), (w_1, w_3), (w_2, w_3)\}$ be the set of proper subsets of W.
7. Define NFSs as

$$\varsigma_{R1}(w_1, w_3) = \{\langle w_1, .3, .4, .5\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .1, .2, .3\rangle, \langle w_{1,2}, .5, .6, .3\rangle, \langle w_{1,3}, .3, .4, .3\rangle, \langle w_{2,3}, .5, .6, .3\rangle\},$$

$$\varsigma_{R2}(w_2, w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .6, .8, .3\rangle, \langle w_3, .4, .5, .2\rangle, \langle w_{1,2}, .6, .8, .2\rangle, \langle w_{1,3}, .4, .5, .2\rangle, \langle w_{2,3}, .6, .8, .2\rangle\} \text{ and}$$

$$\varsigma_{R3}(w_1, w_2) = \{\langle w_1, .4, .5, .2\rangle, \langle w_2, .6, .8, .3\rangle, \langle w_3, 0, 0, 1\rangle, \langle w_{1,2}, .6, .8, .2\rangle, \langle w_{1,3}, .4, .5, .2\rangle, \langle w_{2,3}, .6, .8, .2\rangle\}.$$

8. Then the neutro-fine collection ${}^f\varsigma_W = \{0_n, 1_n, \varsigma_{R1}(w_1, w_3), \varsigma_{R2}(w_2, w_3), \varsigma_{R3}(w_1, w_2)\}$, whose elements are NFOSs. Thus $\left(W, \tau_n, {}^f\varsigma_W\right)$ is a NFTS over (W, τ_n).
9. Let the complement and absolute complement of ${}^f\varsigma_W$ represents the secureness of the items.

The complement of ${}^f\varsigma_W$ is ${}^F\varsigma_W = \{0_n, 1_n, \varsigma_{R1}(w_1, w_3)', \varsigma_{R2}(w_2, w_3)', \varsigma_{R3}(w_1, w_2)'\}$, where

$$\varsigma_{R1}(w_1, w_3)' = \{\langle w_1, .5, .6, .3\rangle, \langle w_2, 1, 1, 0\rangle, \langle w_3, .3, .8, .1\rangle, \langle w_{1,2}, .3, .4, .5\rangle, \langle w_{1,3}, .3, .6, .3\rangle, \langle w_{2,3}, .3, .4, .5\rangle\},$$

$$\varsigma_{R2}(w_2, w_3)' = \{\langle w_1, 1, 1, 0\rangle, \langle w_2, .3, .2, .6\rangle, \langle w_3, .2, .5, .4\rangle, \langle w_{1,2}, .2, .2, .6\rangle, \langle w_{1,3}, .2, .5, .4\rangle, \langle w_{2,3}, .2, .2, .6\rangle\} \text{ and}$$

$$\varsigma_{R3}(w_1, w_2)' = \{\langle w_1, 2, .5, .4\rangle, \langle w_2, .3, .2, .6\rangle, \langle w_3, .1, 1, 0\rangle, \langle w_{1,2}, .2, .2, .6\rangle, \langle w_{1,3}, .2, .5, .4\rangle, \langle w_{2,3}, .2, .2, .6\rangle\}.$$

The absolute complement of ${}^f\varsigma_W$ is ${}^{f*}\varsigma_W = \{0_n, 1_n, \varsigma_{R1}{}^*(w_2), \varsigma_{R2}{}^*(w_1), \varsigma_{R3}{}^*(w_3)\}$, where

$$\varsigma_{R1}{}^*(w_2) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .5, .6, .3\rangle, \langle w_3, 0, 0, 1\rangle, \langle w_{1,2}, .5, .6, .3\rangle, \langle w_{1,3}, .0, 0, 1\rangle, \langle w_{2,3}, .5, .6, .3\rangle\},$$

$$\varsigma_{R2}{}^*(w_1, w_2) = \{\langle w_1, .3, .5, .8\rangle, \langle w_2, .4, .8, .5\rangle, \langle w_3, 0, 0, 1\rangle, \langle w_{1,2}, .4, .8, .5\rangle, \langle w_{1,3}, .3, .5, .7\rangle, \langle w_{2,3}, .4, .8, .5\rangle\} \text{ and}$$

$$\varsigma_{R3}{}^*(w_1, w_3) = \{\langle w_1, .6, .5, .7\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .3, .4, .1\rangle, \langle w_{1,2}, .6, .8, .5\rangle, \langle w_{1,3}, .6, .5, .1\rangle, \langle w_{2,3}, .5, .8, .1\rangle\}.$$

10. By using the formula (5.1), the following values are obtained.

NV(R1) = 1.26,
NV(R2) = **1.62** and
NV(R3) = **1.62**.

11. Thus both NV(R2) and NV(R3) are the highest value.

In this situation, replace the clogged coils (w_2) by some other risk factors and repeat the process.

1. Let $R1$, $R2$, and $R3$ be sample refrigerators for second-hand sale.
2. Let $W = \{w_1, w_2, w_3\}$ be the universe, where w_1 –locked compressor, w_2 –failed fan motor and w_3 –dirty condenser coil.
3. Again analyze the damages on each refrigerator.

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4. Define $R1$, $R2$, and $R3$ as NSs.

$R1=\{\langle w_1,.3,.4,.5\rangle,\langle w_2,.4,.8,.3\rangle,\langle w_3,.1,.2,.3\rangle\}$,

$R2=\{\langle w_1,.4,.5,.2\rangle,\langle w_2,.5,.8,.1\rangle,\langle w_3,.4,.5,.2\rangle\}$ and

$R3=\{\langle w_1,.4,.5,.2\rangle,\langle w_2,.5,.8,.1\rangle,\langle w_3,.4,.5,.2\rangle\}$.

5. Thus $\tau_\eta=\{0_n,1_n,R1,R2,R3\}$ is a NT and so (W,τ_n) is a NTS.

6. Let $V=\{(w_1),(w_2),(w_3),(w_1,w_2),(w_1,w_3),(w_2,w_3)\}$ be the set of proper subsets of W.

7. Define NFSs as

$\varsigma_{R1}(w_2,w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,.4,.8,.3\rangle,\langle w_3,.1,.2,.3\rangle,\langle w_{1,2},.4,.8,.3\rangle,\langle w_{1,3},.3,.4,.3\rangle,\langle w_{2,3},.4,.8,.3\rangle\}$,

$\varsigma_{R2}(w_2)=\{\langle w_1,0,0,1\rangle,\langle w_2,.5,.8,.1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.5,.8,.1\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.5,.8,.1\rangle\}$ and

$\varsigma_{R3}(w_1,w_3)=\{\langle w_1,.4,.5,.2\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.4,.5,.2\rangle,\langle w_{1,2},.5,.8,.1\rangle,\langle w_{1,3},.5,.8,.1\rangle,\langle w_{2,3},.5,.8,.1\rangle\}$.

8. Then the neutro-fine collection ${}^f\varsigma_W=\{0_n,1_n,\varsigma_{R1}(w_2,w_3),\varsigma_{R2}(w_2),\varsigma_{R3}(w_1,w_3)\}$, whose elements are NFOSs. Thus $(W,\tau_n,{}^f\varsigma_W)$ is a NFTS over (W,τ_n).

9. Let the complement and absolute complement of ${}^f\varsigma_W$ represents the secureness of the items.

The complement of ${}^f\varsigma_W$ is ${}^F\varsigma_W=\{0_n,1_n,\varsigma_{R1}(w_2,w_3)',\varsigma_{R2}(w_2)',\varsigma_{R3}(w_1,w_3)'\}$, where

$\varsigma_{R1}(w_2,w_3)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.3,.2,.4\rangle,\langle w_3,.3,.8,.1\rangle,\langle w_{1,2},.3,.2,.4\rangle,\langle w_{1,3},.3,.6,.3\rangle,\langle w_{2,3},.3,.2,.4\rangle\}$,

$\varsigma_{R2}(w_2)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.1,.2,.5\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.1,.2,.5\rangle,\langle w_{1,3},1,1,0\rangle,\langle w_{2,3},.1,.2,.5\rangle\}$ and

$\varsigma_{R3}(w_1,w_3)'=\{\langle w_1,.2,.5,.4\rangle,\langle w_2,1,1,0\rangle,\langle w_3,.2,.5,.4\rangle,\langle w_{1,2},.1,.2,.5\rangle,\langle w_{1,3},.1,.2,.5\rangle,\langle w_{2,3},.1,.2,.5\rangle\}$.

The absolute complement of ${}^f\varsigma_W$ is ${}^{f*}\varsigma_W=\{0_n,1_n,\varsigma_{R1}{}^*(w_1),\varsigma_{R2}{}^*(w_1,w_3),\varsigma_{R3}{}^*(w_2)\}$, where

$\varsigma_{R1}{}^*(w_1)=\{\langle w_1,.3,.4,.5\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.4,.8,.3\rangle,\langle w_{1,3},.3,.4,.3\rangle,\langle w_{2,3},0,0,1\rangle\}$,

$\varsigma_{R2}{}^*(w_1,w_3)=\{\langle w_1,.4,.5,.2\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.4,.5,.2\rangle,\langle w_{1,2},.5,.8,.1\rangle,\langle w_{1,3},.5,.8,.1\rangle,\langle w_{2,3},.5,.8,.1\rangle\}$ and

$\varsigma_{R3}{}^*(w_2)=\{\langle w_1,0,0,1\rangle,\langle w_2,.5,.8,.1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.5,.8,.1\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.5,.8,.1\rangle\}$.

10. By using the formula (5.1), the following values are obtained.

NV(R1) = 1.43,

NV(R2) = **3.00** and

NV(R3) = 1.8.

11. Thus NV(R2) is the highest value.

Hence the customer can prefer to buy R2 for the second use.

6. Conclusions

The principal concern of this paper are to initiate the new type of topology called NFT and studied some essential theorems. Also defined the interior and closure on NFTS, and analyzed its basic properties with perfect examples. The concept of separation axioms on this space are investigated and the relationship between each neutro-fine $T_{i=0,1,2}$-spaces are exposed with illustrative examples. Besides this, the application of this space are explored on DM problems where the complement and absolute complement of each NFOS is determined to change unfavorable queries into a favorable one. The algorithm specified to describe the process and the positive solution calculated by the formula are given. Consequently, the future researchers can extend this NFTS to some special types of sets, whereas soft sets, rough sets, crisp sets, cubic sets, etc. Also, the application part can extend to MCDM problems.

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Neutrosophic Vague Incidence Graph

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Abstract

Vague sets gives more intuitive graphical notation of vague data, that devotes better analysis in information relationships, incompleteness and similarity measures. Neutrosophic graphs are used as a mathematical tool to kept an imprecise and unspecified information. In this paper, the neutrosophic vague incidence graphs are introduced. The edge-connectivity, vertex-connectivity and pair-connectivity in neutrosophic vague incidence graphs are established. The given results are illustrated with suitable example.
Keywords: Neutrosophic vague incidence graph, Edge-connectivity, Vertex-connectivity and Pair-connectivity.

1 Introduction

Vague sets are denoted as a higher-order fuzzy sets which develops the solution procedure more complex to obtain the results more accurate than fuzzy but not affecting the complexity on computation time/volume and memory space. The restrictions in vague sets allow only to hold an incomplete data, but the handling of indeterminate information still remains. Can we see an instance, suppose there are 10 patients to check a pandemic during testing. In that time, there are five patients having positive, three will have negative and two are undecided or yet to come. By employing the neutrosophic concepts, it can be expressed as $x(0.5, 0.2, 0.3)$. Hence the neutrosophic field arises to hold the indeterminacy data. It generalizes the aforementioned sets from the philosophical viewpoint. The single-valued neutrosophic set is the generalisation of intuitionistic fuzzy sets and is used expediently to deal with real-world problems, especially in decision support.[1–5,20,21,40] The computation of believe in that element (truth), the disbelieve in that element (falsehood) and the indeterminacy part of that element with the sum of these three components are strictly less than 1. Neutrosophic sets are the base of neutrosophic logic, a multiple value logic that generalizes the fuzzy logic which deals with paradoxes, contradictions, antitheses, antinomies is proposed by Smarandache[34,36] and references therein.

The neutrosophic set is introduced by the author Smarandache in order to use the inconsistent and indeterminate information, and has been studied extensively (see[34–40]). In the definition of neutrosophic set, the indeterminacy value is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are defined completely independent with the sum of these values lies between 0 and 3. Neutrosophic set and related notions paid attention by the researchers in many weird domains.[9,10] The combination of neutrosophic set and vague set are introduced by Alkhazaleh in 2015.[11] Single valued neutrosophic graph are established in the papers.[16,17] Some types of neutrosophic graphs and co-neutrosophic graphs are discussed in.[23] Intuitionistic bipolar neutrosophic set and its application to graphs are established in.[31] Al-Quran and Hassan in[8] introduced a combination of neutrosophic vague set and soft expert set to improving the reason-ability of decision making in real life application. Neutrosophic vague graphs are investigated in.[30] Comparative study of regular and (highly) irregular vague graphs with applications are obtained in.[18] Furthermore, some properties of degree of vague graphs, domination number and regularity properties of vague graphs are established by the author Borzooei.[12–14] Authors in[7] presented some properties of single-valued neutrosophic incidence graphs and discussed the edge-connectivity, vertex-connectivity and pair-connectivity in neutrosophic incidence graphs. Motivated by papers,[7,11,28,30] we introduce the concept of neutrosophic

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vague incidence graphs. The main contributions of this paper are to introduce the neutrosophic vague incidence graphs, and the edge-connectivity, vertex-connectivity and pair-connectivity in neutrosophic vague incidence graphs are discussed in Section 3.

2 Preliminaries

In this section, basic definitions and example are given.

Definition 2.1. [41] A vague set $\mathbb{A}$ on a non empty set $\mathbb{X}$ is a pair $(T_{\mathbb{A}}, F_{\mathbb{A}})$, where $T_{\mathbb{A}} : \mathbb{X} \to [0,1]$ and $F_{\mathbb{A}} : \mathbb{X} \to [0,1]$ are true membership and false membership functions, respectively, such that

$$0 \leq T_{\mathbb{A}}(x) + F_{\mathbb{A}}(y) \leq 1 \text{ for any } x \in \mathbb{X}.$$

Let $\mathbb{X}$ and $\mathbb{Y}$ be two non-empty sets. A vague relation $\mathbb{R}$ of $\mathbb{X}$ to $\mathbb{Y}$ is a vague set $\mathbb{R}$ on $\mathbb{X} \times \mathbb{Y}$ that is $\mathbb{R} = (T_{\mathbb{R}}, F_{\mathbb{R}})$, where $T_{\mathbb{R}} : \mathbb{X} \times \mathbb{Y} \to [0,1]$, $F_{\mathbb{R}} : \mathbb{X} \times \mathbb{Y} \to [0,1]$ and satisfy the condition:

$$0 \leq T_{\mathbb{R}}(x,y) + F_{\mathbb{R}}(x,y) \leq 1 \text{ for any } x \in \mathbb{X}.$$

Definition 2.2. [12] Let $\mathbb{G}^* = (\mathbb{V}, \mathbb{E})$ be a graph. A pair $\mathbb{G} = (\mathbb{J}, \mathbb{K})$ is called a vague graph on $\mathbb{G}^*$, where $\mathbb{J} = (T_{\mathbb{J}}, F_{\mathbb{J}})$ is a vague set on $\mathbb{V}$ and $\mathbb{K} = (T_{\mathbb{K}}, F_{\mathbb{K}})$ is a vague set on $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ such that for each $xy \in \mathbb{E}$,

$$T_{\mathbb{K}}(xy) \leq \min(T_{\mathbb{J}}(x), T_{\mathbb{J}}(y)) \text{ and } F_{\mathbb{K}}(xy) \geq \max(F_{\mathbb{J}}(x), F_{\mathbb{J}}(y)).$$

Definition 2.3. [19,34] Let $\mathbb{X}$ be a space of points (objects), with a generic elements in $\mathbb{X}$ denoted by x. A single valued neutrosophic set $\mathbb{A}$ in $\mathbb{X}$ is characterised by truth-membership function $T_{\mathbb{A}}(x)$, indeterminacy-membership function $I_{\mathbb{A}}(x)$ and falsity-membership-function $F_{\mathbb{A}}(x)$, For each point x in $\mathbb{X}$, $T_{\mathbb{A}}(x), F_{\mathbb{A}}(x), I_{\mathbb{A}}(x) \in [0,1]$. Also

$$A = \{x, T_{\mathbb{A}}(x), F_{\mathbb{A}}(x), I_{\mathbb{A}}(x)\} \text{ and } 0 \leq T_{\mathbb{A}}(x) + I_{\mathbb{A}}(x) + F_{\mathbb{A}}(x) \leq 3.$$

Definition 2.4. [34] A Neutrosophic set $\mathbb{A}$ is contained in another neutrosophic set $\mathbb{B}$, (i.e) $\mathbb{A} \subseteq \mathbb{B}$ if $\forall x \in \mathbb{X}, T_{\mathbb{A}}(x) \leq T_{\mathbb{B}}(x), I_{\mathbb{A}}(x) \geq I_{\mathbb{B}}(x)$and $F_{\mathbb{A}}(x) \geq F_{\mathbb{B}}(x)$.

Definition 2.5. [6,17] A neutrosophic graph is defined as a pair $\mathbb{G}^* = (\mathbb{V}, \mathbb{E})$ where
(i) $\mathbb{V} = \{v_1, v_2, .., v_n\}$ such that $T_1 : \mathbb{V} \to [0,1]$, $I_1 : \mathbb{V} \to [0,1]$ and $F_1 : \mathbb{V} \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \leq T_1(v) + I_1(v) + F_1(v) \leq 3,$$

(ii) $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ where $T_2 : \mathbb{E} \to [0,1]$, $I_2 : \mathbb{E} \to [0,1]$ and $F_2 : \mathbb{E} \to [0,1]$ are such that

$$\begin{gathered} T_2(uv) \leq \min\{T_1(u), T_1(v)\}, \\ I_2(uv) \leq \min\{I_1(u), I_1(v)\}, \\ F_2(uv) \leq \max\{F_1(u), F_1(v)\}, \\ \text{and } 0 \leq T_2(uv) + I_2(uv) + F_2(uv) \leq 3, \quad \forall uv \in \mathbb{E}. \end{gathered}$$

Definition 2.6. [11] A neutrosophic vague set $\mathbb{A}_{NV}$ (NVS in short) on the universe of discourse $\mathbb{X}$ written as

$$\mathbb{A}_{NV} = \{\langle x, \hat{T}_{\mathbb{A}_{NV}}(x), \hat{I}_{\mathbb{A}_{NV}}(x), \hat{F}_{\mathbb{A}_{NV}}(x)\rangle, x \in \mathbb{X}\},$$

whose truth-membership, indeterminacy membership and falsity-membership function is defined as

$$\hat{T}_{\mathbb{A}_{NV}}(x) = [T^-(x), T^+(x)], \hat{I}_{\mathbb{A}_{NV}}(x) = [I^-(x), I^+(x)] \text{ and } \hat{F}_{\mathbb{A}_{NV}}(x) = [F^-(x), F^+(x)],$$

where $T^+(x) = 1 - F^-(x), F^+(x) = 1 - T^-(x)$, and $0 \leq T^-(x) + I^-(x) + F^-(x) \leq 2$.

Definition 2.7. [11] The complement of NVS $\mathbb{A}_{NV}$ is denoted by $\mathbb{A}^c_{NV}$ and it is defined by

$$\begin{gathered} \hat{T}^c_{\mathbb{A}_{NV}}(x) = [1 - T^+(x), 1 - T^-(x)], \\ \hat{I}^c_{\mathbb{A}_{NV}}(x) = [1 - I^+(x), 1 - I^-(x)], \\ \hat{F}^c_{\mathbb{A}_{NV}}(x) = [1 - F^+(x), 1 - F^-(x)]. \end{gathered}$$

Definition 2.8. [11] Let $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ be two NVSs of the universe $\mathbb{U}$. If for all $u_i \in \mathbb{U}$,

$$\hat{T}_{\mathbb{A}_{NV}}(u_i) \leq \hat{T}_{\mathbb{B}_{NV}}(u_i), \hat{I}_{\mathbb{A}_{NV}}(u_i) \geq \hat{I}_{\mathbb{B}_{NV}}(u_i), \hat{F}_{\mathbb{A}_{NV}}(u_i) \geq \hat{F}_{\mathbb{B}_{NV}}(u_i),$$

then the NVS, $\mathbb{A}_{NV}$ are included in $\mathbb{B}_{NV}$, denoted by $\mathbb{A}_{NV} \subseteq \mathbb{B}_{NV}$ where $1 \leq i \leq n$.

Definition 2.9. [11] The union of two NVSs $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ is a NVSs, $\mathbb{C}_{NV}$, written as $\mathbb{C}_{NV} = \mathbb{A}_{NV} \cup \mathbb{B}_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ by

$$\hat{T}_{\mathbb{C}_{NV}}(x) = [\max(T^-_{\mathbb{A}_{NV}}(x), T^-_{\mathbb{B}_{NV}}(x)), \max(T^+_{\mathbb{A}_{NV}}(x), T^+_{\mathbb{B}_{NV}}(x))]$$
$$\hat{I}_{\mathbb{C}_{NV}}(x) = [\min(I^-_{\mathbb{A}_{NV}}(x), I^-_{\mathbb{B}_{NV}}(x)), \min(I^+_{\mathbb{A}_{NV}}(x), I^+_{\mathbb{B}_{NV}}(x))]$$
$$\hat{F}_{\mathbb{C}_{NV}}(x) = [\min(F^-_{\mathbb{A}_{NV}}(x), F^-_{\mathbb{B}_{NV}}(x)), \min(F^+_{\mathbb{A}_{NV}}(x), F^+_{\mathbb{B}_{NV}}(x))].$$

Definition 2.10. [11] The intersection of two NVSs, A_{NV} and B_{NV} is a NVSs C_{NV}, written as $C_{NV} = A_{NV} \cap B_{NV}$, whose truth-membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by

$$\hat{T}_{\mathbb{C}_{NV}}(x) = [\min(T^-_{\mathbb{A}_{NV}}(x), T^-_{\mathbb{B}_{NV}}(x)), \min(T^+_{\mathbb{A}_{NV}}(x), T^+_{\mathbb{B}_{NV}}(x))]$$
$$\hat{I}_{\mathbb{C}_{NV}}(x) = [\max(I^-_{\mathbb{A}_{NV}}(x), I^-_{\mathbb{B}_{NV}}(x)), \max(I^+_{\mathbb{A}_{NV}}(x), I^+_{\mathbb{B}_{NV}}(x))]$$
$$\hat{F}_{\mathbb{C}_{NV}}(x) = [\max(F^-_{\mathbb{A}_{NV}}(x), F^-_{\mathbb{B}_{NV}}(x)), \max(F^+_{\mathbb{A}_{NV}}(x), F^+_{\mathbb{B}_{NV}}(x))].$$

Definition 2.11. [30] Let $\mathbb{G}^* = (\mathbb{R}, \mathbb{S})$ be a graph. A pair $\mathbb{G} = (\mathbb{A}, \mathbb{B})$ is called a neutrosophic vague graph (NVG) on $\mathbb{G}^*$ or a neutrosophic vague graph where $\mathbb{A} = (\hat{T}_{\mathbb{A}}, \hat{I}_{\mathbb{A}}, \hat{F}_{\mathbb{A}})$ is a neutrosophic vague set on $\mathbb{R}$ and $\mathbb{B} = (\hat{T}_{\mathbb{B}}, \hat{I}_{\mathbb{B}}, \hat{F}_{\mathbb{B}})$ is a neutrosophic vague set $\mathbb{S} \subseteq \mathbb{R} \times \mathbb{R}$ where

(1)$\mathbb{R} = \{v_1, v_2, ..., v_n\}$ such that $T^-_{\mathbb{A}} : \mathbb{R} \to [0,1], I^-_{\mathbb{A}} : \mathbb{R} \to [0,1], F^-_{\mathbb{A}} : \mathbb{R} \to [0,1]$ which satisfies the condition $F^-_{\mathbb{A}} = [1 - T^+_{\mathbb{A}}]$
$T^+_{\mathbb{A}} : \mathbb{R} \to [0,1], I^+_{\mathbb{A}} : \mathbb{R} \to [0,1], F^+_{\mathbb{A}} : \mathbb{R} \to [0,1]$ which satisfies the condition $F^+_{\mathbb{A}} = [1 - T^-_{\mathbb{A}}]$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i \in \mathbb{R}$, and

$$0 \leq T^-_{\mathbb{A}}(v_i) + I^-_{\mathbb{A}}(v_i) + F^-_{\mathbb{A}}(v_i) \leq 2$$
$$0 \leq T^+_{\mathbb{A}}(v_i) + I^+_{\mathbb{A}}(v_i) + F^+_{\mathbb{A}}(v_i) \leq 2.$$

(2) $\mathbb{S} \subseteq \mathbb{R} \times \mathbb{R}$ where

$$T^-_{\mathbb{B}} : \mathbb{R} \times \mathbb{R} \to [0,1], I^-_{\mathbb{B}} : \mathbb{R} \times \mathbb{R} \to [0,1], F^-_{\mathbb{B}} : \mathbb{R} \times \mathbb{R} \to [0,1]$$
$$T^+_{\mathbb{B}} : \mathbb{R} \times \mathbb{R} \to [0,1], I^+_{\mathbb{B}} : \mathbb{R} \times \mathbb{R} \to [0,1], F^+_{\mathbb{B}} : \mathbb{R} \times \mathbb{R} \to [0,1]$$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i, v_j \in \mathbb{S}$, respectively and such that,

$$0 \leq T^-_{\mathbb{B}}(v_iv_j) + I^-_{\mathbb{B}}(v_iv_j) + F^-_{\mathbb{B}}(v_iv_j) \leq 2$$
$$0 \leq T^+_{\mathbb{B}}(v_iv_j) + I^+_{\mathbb{B}}(v_iv_j) + F^+_{\mathbb{B}}(v_iv_j) \leq 2,$$

such that

$$T^-_{\mathbb{B}}(v_iv_j) \leq \min\{T^-_{\mathbb{A}}(v_i), T^-_{\mathbb{A}}(v_j)\}$$
$$I^-_{\mathbb{B}}(v_iv_j) \leq \min\{I^-_{\mathbb{A}}(v_i), I^-_{\mathbb{A}}(v_j)\}$$
$$F^-_{\mathbb{B}}(v_iv_j) \leq \max\{F^-_{\mathbb{A}}(v_i), F^-_{\mathbb{A}}(v_j)\},$$

and similarly

$$T^+_{\mathbb{B}}(v_iv_j) \leq \min\{T^+_{\mathbb{A}}(v_i), T^+_{\mathbb{A}}(v_j)\}$$
$$I^+_{\mathbb{B}}(v_iv_j) \leq \min\{I^+_{\mathbb{A}}(v_i), I^+_{\mathbb{A}}(v_j)\}$$
$$F^+_{\mathbb{B}}(v_iv_j) \leq \max\{F^+_{\mathbb{A}}(v_i), F^+_{\mathbb{A}}(v_j)\}.$$

Definition 2.12. [7] A neutrosophic incidence graph of an incidence graph, $G^* = (V, E, I)$, is an ordered triplet, $\tilde{G} = (A, B, C)$, such that
1. A is a neutrosophic set on V,
2. B is a neutrosophic relation on V and
3. C is a neutrosophic subset of $V \times E$ such that

$$\begin{aligned} T_C(x, xy) &\leq \min\{T_A(x), T_B(xy)\}, \\ I_C(x, xy) &\leq \min\{I_A(x), I_B(xy)\}, \\ F_C(x, xy) &\leq \max\{F_A(x), F_B(xy)\}, \text{ for all } xy \in E \end{aligned}$$

3 Neutrosophic Vague incidence graph Graphs

In this section, the definition of NVIGs are introduced. Some properties on edge-connectivity, vertex-connectivity and pair-connectivity in neutrosophic vague incidence graphs are established.

Definition 3.1. A neutrosophic vague incidence graph of an incidence graph $G = (\mathbb{V}, \mathbb{E}, \mathbb{I})$, is an ordered triplet, $G^* = (Q, R, S)$, such that
1. Q is a neutrosophic vague set on $\mathbb{V}$,
2. R is a neutrosophic vague relation on $\mathbb{V}$ and
3. S is a neutrosophic vague subset of $\mathbb{V} \times \mathbb{E}$ such that

$$\begin{aligned} T_S^-(a, ab) &\leq \min\{T_Q^-(a), T_R^-(ab)\}, \\ I_S^-(a, ab) &\leq \min\{I_Q^-(a), I_R^-(ab)\}, \\ F_S^-(a, ab) &\leq \max\{F_Q^-(a), F_R^-(ab)\}, \end{aligned}$$

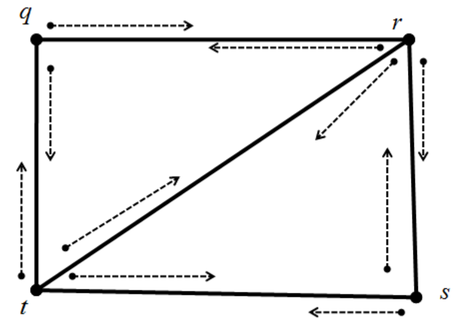
similarly

$$\begin{aligned} T_S^+(a, ab) &\leq \min\{T_Q^+(a), T_R^+(ab)\}, \\ I_S^+(a, ab) &\leq \min\{I_Q^+(a), I_R^+(ab)\}, \\ F_S^+(a, ab) &\leq \max\{F_Q^+(a), F_R^+(ab)\}, \forall a \in \mathbb{V}, ab \in \mathbb{E}. \end{aligned}$$

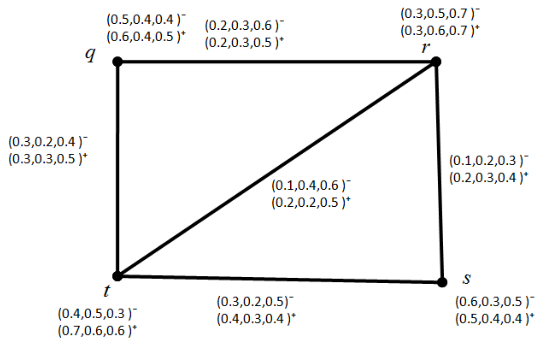
Example 3.2. Consider an incidence graph $G = (\mathbb{V}, \mathbb{E}, \mathbb{I})$ such that $\mathbb{V} = \{q, r, s, t\}$, $\mathbb{E} = \{qr, rs, rt, st, qt\}$ and $\mathbb{I} = \{(q, qr), (r, qr), (r, rs), (s, rs), (r, rt), (t, rt), (s, st), (t, st), (q, qt), (t, qt)\}$, as shown in figure 1
Let $G^* = (Q, R, S)$ be a neutrosophic vague incidence graph associated with G, as shown in figure 3, where $q = [0.5, 0.6], [0.4, 0.4], [0.4, 0.5]$, $r = [0.3, 0.3], [0.5, 0.6], [0.7, 0.7]$, $s = [0.6, 0.5], [0.3, 0.4], [0.5, 0.4]$, $t = [0.4, 0.7], [0.5, 0.6], [0.3, 0.6]$
$q^- = (0.5, 0.4, 0.4), q^+ = (0.6, 0.4, 0.5)$ $r^- = (0.3, 0.5, 0.7), r^+ = (0.3, 0.6, 0.7)$
$s^- = (0.6, 0.3, 0.5), q^+ = (0.5, 0.4, 0.4)$ $t^- = (0.3, 0.6, 0.7), q^+ = (0.7, 0.6, 0.6)$

$$\begin{aligned} &Q = \{q^- = (0.5, 0.4, 0.4), q^+ = (0.6, 0.4, 0.5), r^- = (0.3, 0.5, 0.7), r^+ = (0.3, 0.6, 0.7), \\ &\quad s^- = (0.6, 0.3, 0.5), s^+ = (0.5, 0.4, 0.4), t^- = (0.3, 0.6, 0.7), t^+ = (0.7, 0.6, 0.6)\} \\ &R = \{(qr)^- = (0.2, 0.3, 0.6), (qr)^+ = (0.2, 0.3, 0.5), (st)^- = (0.3, 0.2, 0.5), (st)^+ = (0.4, 0.3, 0.4), \\ &\quad (rs)^+ = (0.1, 0.2, 0.3), (rs)^+ = (0.2, 0.3, 0.4)(qt)^- = (0.3, 0.2, 0.4), (qt)^+ = (0.3, 0.3, 0.5), \\ &\quad (rt)^- = (0.1, 0.4, 0.6), (rt)^+ = (0.2, 0.2, 0.5)\} \\ &S = \{(q, qr)^- = (0.2, 0.2, 0.5), (q, qr)^+ = (0.2, 0.1, 0.4), (r, qr)^- = (0.1, 0.3, 0.5), (r, qr)^+ = (0.1, 0.2, 0.4), \\ &\quad (r, rs)^- = (0.1, 0.2, 0.6), (r, rs)^+ = (0.1, 0.3, 0.4), (s, rs)^- = (0.1, 0.2, 0.5), (s, rs)^+ = (0.1, 0.3, 0.3), \\ &\quad (r, rt)^- = (0.1, 0.3, 0.6), (r, rt)^+ = (0.2, 0.1, 0.6), (t, rt)^- = (0.1, 0.3, 0.5), (t, rt)^+ = (0.2, 0.1, 0.5), \\ &\quad (s, st)^- = (0.2, 0.1, 0.4), (s, st)^+ = (0.3, 0.2, 0.3)(t, st)^- = (0.2, 0.1, 0.4), (t, st)^+ = (0.3, 0.2, 0.3), \\ &\quad (q, qt)^- = (0.2, 0.1, 0.3), (q, qt)^+ = (0.2, 0.2, 0.4), (t, qt)^- = (0.2, 0.2, 0.3), (t, qt)^+ = (0.2, 0.2, 0.5)\} \end{aligned}$$

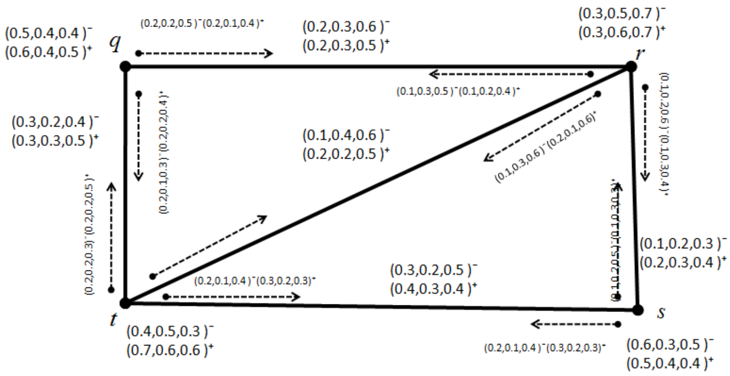
Definition 3.3. The support of an NVIG $G^* = (Q, R, S)$ is denoted by $G^{**} = (Q^*, R^*, S^*)$ where
$Q^* =$ support of $Q = \{a \in \mathbb{V} : \hat{T}_Q(a) > 0, \hat{I}_Q(a) > 0, \hat{F}_Q(a) > 0\}$
$R^* =$ support of $R = \{ab \in \mathbb{E} : \hat{T}_R(ab) > 0, \hat{I}_R(ab) > 0, \hat{F}_R(ab) > 0\}$
$S^* =$ support of $S = \{(a, ab) \in \mathbb{I} : \hat{T}_S(a, ab) > 0, \hat{I}_S(a, ab) > 0, \hat{F}_S(a, ab) > 0\}$.



$Figure\,1$
Incidence graph



$Figure\,2$
Neutrosophic Vague Graph



$Figure\,3$
Neutrosophic Vague Incidence Graph

Definition 3.4. If $ab \in R^*$, then ab is the edges of the NVIG $G^* = (Q, R, S)$ and if $(a, ab), (b, ab) \in S^*$ then (a, ab) and (b, ab) are called pair of G^*.

Definition 3.5. Suppose $P = a_0, (a_0, a_0a_1), a_0a_1, (a_1, a_0, a_1), a_1, (a_1, a_1a_2), a_1a_2, (a_2, a_1, a_2), ...a_{n-1}, (a_{n-1}, a_{n-1}a_n), a_{n-1}a_n, (a_n, a_{n-1}, a_n)$ of vertices, edges and pairs in G^* is a walk. It is a closed walk if $a_0 = a_n$. In the above sequence, if all edges are distinct, then it is trail, and if the pairs are distinct, then it is an incidence trail. P is called a path, if the vertices are distinct. A path is called a cycle if the initial and end vertices of the path are same. Any two vertices of G^* are said to be connected, if they are joined by a path.

Example 3.6. In the above example, one can see that
$P_1 = q,(q,qr),qr,(r,qr),r,(r,rs),rs,(s,rs),t,(t,tq),tq,(q,tq),q$ is a walk. It is a closed walk since the initial and final vertices are same. (i.e) it is not a path, but it is a trail and incidence trail
$P_2 = q,(q,qr),qr,(r,qr),r,(r,rs),rt,(t,rt),t$. Then, P_2 is a walk, path trail and incidence trail.

Definition 3.7. Let $G^* = (Q,R,S)$ be an NVIG, then $H = (L,M,N)$ is a neutrosophic vague incidence subgraph of G^*, if $L \subseteq Q$, $M \subseteq R$ and $N \subseteq S$. Also, H is a neutrsophic incidence spanning subgraph of G^*, if $L = Q$.

Definition 3.8. In an NVIG, the strength of a path, P is an ordered triplet denoted by $\mathbb{S}(P) = (s_1,s_2,s_3)$, where

$$s_1 = \min\{\hat{T}_R(ab) : ab \in P\}, s_2 = \min\{\hat{I}_R(ab) : ab \in P\}, s_3 = \max\{\hat{F}_R(ab) : ab \in P\}.$$

Similarly, the incidence strength of a path, P, in an NVG is denoted by $I\mathbb{S}(P) = (is_1,is_2,is_3)$, where

$$is_1 = \min\{\hat{T}_S(ab) : (a,ab) \in P\}, is_2 = \min\{\hat{I}_S(ab) : (a,ab) \in P\}, is_3 = \max\{\hat{F}_S(ab) : (a,ab) \in P\}.$$

Definition 3.9. In an NVG, $G^* = (Q,R,S)$ the greatest strength of the path from l to m, where $l,m \in Q^* \cup R^*$ is the maximum of strength of all paths from l to m.

$$\begin{aligned}\mathbb{S}^\infty(l,m) &= \max\{\mathbb{S}(P_1),\mathbb{S}(P_2),\mathbb{S}(P_3),...\}\\ &= (s_1^\infty,s_2^\infty,s_3^\infty)\\ &= (\max(s_{11},s_{12},s_{13},....),\max(s_{21},s_{22},s_{23},...),\min(s_{31},s_{32},s_{33},..)),\end{aligned}$$

$\mathbb{S}^\infty(l,m)$ is sometimes called the connectedness between l and m.
Similarly, the greatest incidence strength of the path from l to m, where $l,m \in Q^* \cup R^*$ is the maximum of incidence strength of all paths from l to m.

$$\begin{aligned}I\mathbb{S}^\infty(l,m) &= \max\{I\mathbb{S}(P_1),I\mathbb{S}(P_2),I\mathbb{S}(P_3),...\}\\ &= (is_1^\infty,is_2^\infty,is_3^\infty)\\ &= (\max(is_{11},is_{12},is_{13},....),\max(is_{21},is_{22},is_{23},...),\min(is_{31},is_{32},is_{33},..)),\end{aligned}$$

where $P_j, j = 1,2,3,...$ are different paths from l to m.
$I\mathbb{S}^\infty(l,m)$ is sometimes represented as the incidence connectedness between l to m.

Definition 3.10. An NVG, $G^* = (Q,R,S)$ is a cycle if and only if, the underlying graph, $G^{**} = (Q^*,R^*,S^*)$ is a cycle.

Definition 3.11. An NVG, $G^* = (Q,R,S)$ is a is a neutrosophic vague cycle if and only if, $G^{**} = (Q^*,R^*,S^*)$ is a cycle and there exist no unique edge, $ab \in R^*$ such that

$$\begin{aligned}\hat{T}_R(xy) &= \min\{\hat{T}_R(ab) : ab \in R^*\},\\ \hat{I}_R(xy) &= \min\{\hat{I}_R(ab) : ab \in R^*\},\\ \hat{F}_R(xy) &= \max\{\hat{F}_R(ab) : ab \in R^*\}.\end{aligned}$$

Definition 3.12. An NVG, $G^* = (Q,R,S)$ is a neutrosophic vague incidence cycle if and only if, $G^{**} = (Q^*,R^*,S^*)$ is a cycle and there exist no unique edge, $ab \in S^*$ such that

$$\begin{aligned}\hat{T}_S(x,xy) &= \min\{\hat{T}_S(a,ab) : ab \in S^*\},\\ \hat{I}_S(x,xy) &= \min\{\hat{I}_S(a,ab) : ab \in S^*\},\\ \hat{F}_S(x,xy) &= \max\{\hat{F}_S(a,ab) : ab \in S^*\}.\end{aligned}$$

Definition 3.13. Let $G^* = (Q,R,S)$ be an NVIG. An edge ab in G is called a bridge if and only if, ab is a bridge in $G^{**} = (Q^*,R^*,S^*)$ that is, the removal of ab disconnectes G^{**}. An edge, ab is called a neutrosophic vague bridge if

$$\begin{aligned}\mathbb{S}'^\infty(x,y) &< \mathbb{S}^\infty(x,y), \text{ for some } x,y \in Q^*\\ (s_1'^\infty,s_2'^\infty,s_3'^\infty,) &< (s_1^\infty,s_2^\infty,s_3^\infty,)\\ \Rightarrow s_1'^\infty &< s_1^\infty,\ s_2'^\infty < s_2^\infty,\ s_3'^\infty > s_3^\infty,\end{aligned}$$

where $\mathbb{S}'^{\infty}$ and $\mathbb{S}^{\infty}$ denote the connectedness between x and y in $G' = G^* - ab$ respectively.
An edge ab is called a neutrosophic vague incidence bridge if

$$\begin{aligned} I\mathbb{S}'^{\infty}(x,y) < I\mathbb{S}^{\infty}(x,y),\ \text{for some } x,y \in Q^* \\ (is_1'^{\infty}, is_2'^{\infty}, is_3'^{\infty},) < (is_1^{\infty}, is_2^{\infty}, is_3^{\infty},) \\ \Rightarrow is_1'^{\infty} < is_1^{\infty},\ is_2'^{\infty} < is_2^{\infty},\ is_3'^{\infty} > is_3^{\infty}, \end{aligned}$$

where $I\mathbb{S}'^{\infty}$ and $I\mathbb{S}^{\infty}$ denote the connectedness between x and y in $G' = G^* - ab$ respectively.

Definition 3.14. Let $G^* = (Q, R, S)$ be an NVG. A vertex v in G^* is a cutvertex if and only if it is a cutvertex in $G^{**} = (Q^*, R^*, S^*)$ that is $G^* - v$ is a disconnected graph.
A vertex v in an NVIG is called a neutrosophic vague cutvertex if the connectedness between any two vertices in $G' = G^* - v$ is less than the conncetedness between the same vertices in G^* that is,

$$\mathbb{S}'^{\infty}(x,y) < \mathbb{S}^{\infty}(x,y),\ \text{for some } x,y \in Q^*$$

A vertex v in NVIG G^* is a neutrosophic vague incidence cutvertex if for any pair of vertices, x, y other than v the following condition holds:

$$I\mathbb{S}'^{\infty}(x,y) < I\mathbb{S}^{\infty}(x,y),\ \text{for some } x,y \in Q^*$$

where $I\mathbb{S}'^{\infty}$ and $I\mathbb{S}^{\infty}$ denote the connectedness between x and y in $G' = G^* - ab$ respectively.

Definition 3.15. Let $G^* = (Q, R, S)$ be an NVIG. A pair (a, ab) is called a cutpair if and only if, (a, ab) is a cutpair in $G^{**} = (Q^*, R^*, S^*)$ that is after removing the pair (a, ab) there is no path between a and ab. Let $G^* = (Q, R, S)$ be an NVIG. A pair (a,ab) is called a neutrosophic vague cutpair if deleting the pair (a, ab) reduces the connectedness between $a, ab \in Q^* \cup R^*$ that is

$$\mathbb{S}'^{\infty}(a,ab) < \mathbb{S}^{\infty}(a,ab),$$

where $\mathbb{S}'^{\infty}(a,ab)$ and $\mathbb{S}^{\infty}(a,ab)$ denote the connectedness between a and ab in $G' = G^* - \{(a,ab)\}$ and G^* respectively.
A pair (a, ab) is called neutrosophic vague incidence cutpair if

$$I\mathbb{S}'^{\infty}(a,ab) < I\mathbb{S}^{\infty}(a,ab),$$

for $a, ab \in Q^* \cup R^*$
where $I\mathbb{S}'^{\infty}(a,ab)$ and $I\mathbb{S}^{\infty}(a,ab)$ denotes the connectedness between a and ab in $G' = G^* - \{(a,ab)\}$ and G^* respectively.

Theorem 3.16. *Let $G^* = (Q, R, S)$ be a NVIG. If ab is a neutrosophic bridge, then ab is not a weakest edge in any cycle.*

Proof. Let ab be a neutrosophic vague bridge and suppose, on the contrary that ab is the weakest edge of a cycle. Then, in this cycle, we can find an alternative path, P_1 from a to b that does not contain the edge ab and $\mathbb{S}P_1$ is greater than of equal to $\mathbb{S}P_2$, where P_2 is the path involving the edge ab. Thus, removal of the edge ab from G^* does not affect the connectedness between a and $v - a$ contradiction to our assumption. Hence, ab is not the weakest edge in any cycle. □

Theorem 3.17. *If (a, ab) is a neutrosophic vague incidence cutpair, then (a, ab) is not the weakest pair any cycle.*

Proof. Let (a, ab) be a neutrosophic vague incidence cutpair in G^*. On contrary, suppose that (a, ab) is a weakest pair of a cycle. Then we can find an alternative path from a and ab having incidence strength greater than or equal to that of the path involving the pair (a, ab). Thus, removal of the pair (a, ab) does not affect the incidence connectedness between a and ab, but this is a contradiction to our assumption that (a, ab) is a neutrosophic vague incidence cutpair. Hence (a, ab) is not a weakest pair in any cycle. □

Theorem 3.18. *Let $G^* = (Q, R, S)$ be a NVIG. If ab is a neutrosophic vague bridge in G^*, then*

$$\mathbb{S}^{\infty}(a,b) = (s_1^{\infty}, s_2^{\infty}, s_3^{\infty}) = (\hat{T}_R(ab), \hat{I}_R(ab), \hat{F}_R(ab))$$

Proof. Let G^* be an NVIG and ab is a neutrosophic vague bridge in G^*. On the contrary, suppose that

$$\mathbb{S}^{\infty}(a,b) > (\hat{T}_R(ab), \hat{I}_R(ab), \hat{F}_R(ab))$$

Then, there exists a $a-b$ path, P with

$$\mathbb{S}(P) > (\hat{T}_R(ab), \hat{I}_R(ab), \hat{F}_R(ab))$$

and

$$(\hat{T}_R(xy), \hat{I}_R(xy), \hat{F}_R(xy)) > (\hat{T}_R(ab), \hat{I}_R(ab), \hat{F}_R(ab))$$

for all edges on path P. Now, P together with the edge, ab forms a cycle in which ab is the weakest edge, but it is a contradiction to the fact that ab is a neutrosophic vague bridge. Hence

$$\mathbb{S}^{\infty}(a,b) = (s_1^{\infty}, s_2^{\infty}, s_3^{\infty}) = (\hat{T}_R(ab), \hat{I}_R(ab), \hat{F}_R(ab))$$

. □

Theorem 3.19. *If (a, ab) is a neutrosophic vague incidence cutpair in an NVIG $G^* = (Q, R, S)$ then*

$$I\mathbb{S}^{\infty}(a,ab) = (is_1^{\infty}, is_2^{\infty}, is_3^{\infty}) = (\hat{T}_S(a,ab), \hat{I}_S(a,ab), \hat{F}_S(a,ab))$$

.

Proof. The proof is on the same line as the above theorem. □

Theorem 3.20. *Let $G^* = (Q, R, S)$ be an NVIG and G^{**} is a cycle. then an edge ab is a neutrosophic vague bridge of G^* if and only if it is an edge common to two neutrosophic vague incidence cutpairs.*

Proof. Suppose that ab is a neutrosophic vague bridge of G^*. Then there exist vertices a and b with the ab edge lying on every path with the greatest incidence strength between a and b. Consequently, there exists only one path, $P(say)$ between a and b which contains a ab edge and has the greatest incidence strength. Any pair on P will be a neutrosophic vague incidence cutpair, since the removal of any one of them will disconnect P and reduce the incidence strength. Conversely, let ab be an edge common to two neutrosophic vague incidence cutpairs (a, ab) and (b, ab). Thus both (a, ab) and (b, ab) are not the weakest cutpair of G^*. Now, G^{**} being a cycle, there exists only two paths between any two vertices. Also the path P_1 from the vertex a and b not containing the pairs (a, ab) and (b, ab) has less incidence strength than the path containing them. Thus, the path with the greatest incidence strength from a to b is

$$P_2 : a, (a, ab), ab, (b, ab), b.$$

Also,

$$\mathbb{S}^{\infty}(a,b) = \mathbb{S}(P_2) = (\hat{T}_R(ab), \hat{I}_R(ab), \hat{F}_R(ab)).$$

Therefore, ab is a neutrosophic vague bridge. □

4 Conclusion

In this work, the neutrosophic vague incidence graphs have been introduced. The edge-connectivity, vertex-connectivity and pair-connectivity in neutrosophic vague incidence graphs have been established. The given results are illustrated with suitable example. In future, intuitionistic neutrosophic incidence graphs and neutrosophic soft incidence graphs with their properties will be developed.

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Neutrosophic Soft Block Matrices And Some Of Its Properties

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Abstract

In real life situations, there are many issues in which there are uncertainties, vagueness, complexities and unpredictability. Neutrosophic sets are a mathematical tool to address some issues which cannot be met using the existing methods. Neutrosophic soft matrices play a crucial role in handling indeterminant and inconsistent information during decision making process. The main focus of this article is to discuss the concept of neutrosophic sets, neutrosophic soft sets, neutrosophic soft matrices theory and finally to discuss about neutrosophic soft block matrics which are very useful and applicable in various situations involving uncertainties and imprecisions. In this article, neutrosophic soft block matrices, various types of neutrosophic soft block matrices, some operations on it along with some properties associated with it are discussed in details.

Keywords: Fuzzy sets, soft sets, soft matrix, neutrosophic sets, neutrosophic soft sets, neutrosophic soft matrix.

1.Introduction

The theory of fuzzy sets introduced by Zadeh[1], showed applications in many field of studies. This idea of fuzzy sets is welcome because it handles uncertainty and vagueness which cannot be met with classical set theory. Fuzzy set has membership function which assigns to each element of the Universe of discourse, a number from the unit interval [0, 1], to indicate the degree of belongingness of the set under consideration. In the fuzzy set theory, the membership of an element to a fuzzy set is a single value between zero and one. But in reality, it may not always be true that the degree of non membership of an element in a fuzzy set is equal to one minus the membership degree because there may be some hesitation degree as well. Therefore, a generalization of fuzzy set was realized by Atanassv [2], as intuitionistic fuzzy set in which the elements have degrees of membership and non membership which belong to the real unit interval [0,1] and the sum of these two functions belongs to the same interval.

Intuitionistic fuzzy sets as generalization of fuzzy sets is useful in some situation when the description of a problem by linguistic variable, given in terms of membership function only seems to be too difficult to handle. For example, in decision making problem particularly in the case of medical diagnosis, sales analysis, new product marketing, financial services etc, there is a fair chance of a non null hesitation part in each moment of evaluation of an unknown project.

In real life situations, most of the problems in economics, social sciences, environment etc, have various uncertainties. However, most of the existing mathematical tools for formal modeling, reasoning and computation are crisp deterministic and precise in character. There are theories namely, theory of probability, evidence, fuzzy set, intuitionistic fuzzy sets, rough sets etc for dealing with uncertainties.

These theories have their own difficulties as pointed out by Molodtsov[3], and as such the novel concept of soft set theory was initiated. Soft set theory has rich potential for application in solving practical problems in

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economics, social science, medical science etc. Maji *.et.al* ([4], [5]) have studied the theory of fuzzy soft set. Maji. *et. al* [6], have extended the theory of fuzzy soft set to intuitionistic fuzzy soft sets.

Intuitionistic fuzzy sets can only handle the incomplete information considering both the truth membership and falsity membership values. It does not handle the interminants and inconsistants information which exist in belief system.

Smarandache [7], introduced the concept of neutrosiphic sets as a mathematical tool to deal with some situations which involves impreciseness, inconsistencies and interminancy. It is expected that neutrosophic sets will produce more accurate result than those obtained by using fuzzy sets or intuitionistic fuzzy sets. Maji. *et. al* [8], have extended the theory of neutrosophic set to neutrosophic soft set. Later on Maji. *et. al* [9], have used the theory of neutrosophic soft set to decision making process.

Using these concepts, several mathematicians have initiated many research works in different mathematical structures, for instance Deli *et.al* [10, 11,12],. Later, this concept has been modified by Deli and Broumi [13] to develop the idea of neutrosophic soft matrices and its successful utilization in decision making process. Broumi and Smarandache [14] introduced the concept of intuitionistic neutrosophic sets and related properties. Accordingly, Bera and Mahapatra [15] introduce some view on algebraic structure on neutrosophic soft set.

Many researchers extended the concept of neutrosophic soft sets to neutrosophic parameterized soft sets, neutrosophic parameterized soft relations, interval neutrosophic soft sets, interval valued neutrosophic soft sets, interval valued parameterized neutrosophic soft sets, single valued neutrosophic soft sets, linear optimization of single valued neutrosophic soft sets, linear optimization of single valued neutrosophic soft sets, neutrosophic parameterized neutrosophic soft sets, interval valued neutrosophic parameterized interval valued neutrosophic soft sets which can be found in the references ([16],[17], [18],[19],[20], [21], [22]).
Further researches on the extension of neutrosophic soft sets to many other direction are going on and these are visible in valuable works ([23], [24], [25], [26])

In this article, the main aim is to introduce the concept of neutrosophic soft block matrices and thereafter to discuss about various types of neutrosophic block matrices. The transpose of neutrosophic soft block matrix will also be defined. In the process some operations on neutrosophic soft block matrices are defined and accordingly some properties will be discussed.

2. Definition and Preliminaries

Some basic definitions that are useful in subsequent sections of this article are discussed in this section.

Definition 1: Soft set(Molodsov, 1999)

Suppose that U is an initial universe of discourse and E is the set of parameters, let P (U) denote the power set of U. A pair (E, F) is called a soft set over U where F is a mapping given by $F : E \to P(U)$. Clearly soft set is a mapping from parameters to P (U).

Example: Let U= {u_1, u_2, u_3, u_4}be a set of four types of ornaments and E={costly(e_1), Medium(e_2), Cheap(e_3)} be the set of parameters. If $A = \{e_1, e_3\} \subseteq E$. Let F($e_1$)={$u_1$, u_4} and F(e_3)={ u_2, u_3}. Then the soft set can be described as
(F, E)={(e_1,{u_1, u_4}), (e_3, { u_2, u_3})} over U which describes the " Quality of Ornaments" which MR. Z is going to buy.

This soft set can be represented in the following table 1:

U	Costly(e_1)	Medium(e_2)	Cheap(e_3)
u_1	1	0	0
u_2	0	0	1
u_3	0	0	1
u_4	1	0	0

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Table1

Definition 2: Fuzzy soft set (Maji. *et. al.*, 2001)

Suppose that U is an initial universe of discourse and E is the set of parameters. Let $A \subseteq E$. A pair (F_A, E) is called fuzzy soft set over U where F_A is a mapping given by $F_A : E \to P(U)$.where $P(U)$ denotes the collection of all fuzzy subsets of U.

Definition 3: Intuitionistic fuzzy sets (Atanasav, 1986)

Let U be an universe of discourse. Then the intuitionistic fuzzy set A is an object having the form $A = \{< x, \mu_A(x), \nu_A(x) >, x \in U\}$, where the function $\mu_A(x), \nu_A(x) : U \to [0,1]$ define the degree of membership and the degree of non membership of the element $x \in X$ to the set A with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$

Definition 4: Intuitionistic fuzzy soft set (Maji, *et. al*, 2001)

Suppose that U is an initial universe of discourse and E is the set of parameters. Let P(U) denotes the collection of all intuitionistic fuzzy subsets of U. Let $A \subseteq E$. A pair (F_A,E) is called an intuitionistic fuzzy soft set over U where F_A is a mapping given by $F_A : E \to P(U)$

Definition 5: Neutrosophic sets (Smarandache, 2005)

Let U be the universe of discourse, The neutrosophic set A on the universe of discourse U is defined as $A = \{< T_A(x), I_A(x), F_A(x) >: x \in U\}$, where the characteristic functions $T, I, F : U \to [0,1]$ and $^-0 \le T + I + F \le 3^+$; T,I,F are neutrosophic components which defines the degree of membership, the degree of interminancy and the the degree of non membership respectively.

Definition 6: Neutrosophic soft set (Maji, 2013)

Suppose that U is an initial universe set and E is the set of parameters. Let P(U) denotes the collection of all neutrosophic subsets of U. Let $A \subseteq E$. A pair (F_A,E) is called neutrosophic soft set over U where F_A is a mapping given by $F_A : E \to P(U)$

Let us consider the following example for illustration purpose

Let U be the set of houses under consideration and E be the set of parameters where each parameters includes neutrosophic words.

Example: Let U={u_1, u_2, u_3, u_4, u_5,u_6}is a universal set of ornaments and E={ e_1, e_2, e_3, e_4} be the set of parameters where e_1, e_2, e_3, e_4 stands for costly, medium, cheap and very cheap.Let E={ e_1, e_2, e_4} Let us consider the following case:

$F_A(e_1)$={(u_1, 0.3, 0.4,0.2), (u_2, 0.5, 0.4, 0.1),(u_3,0.4, 0.4, 0.2), (u_4, 0.5, 0.2,0.1), (u_5, 0.6, 0.2,0.2), (u_6, 0.5,0.2, 0.2)}
$F_A(e_2)$={(u_1, 0.4, 0.5,0.1), (u_2, 0.4, 0.2,0.3),(u_3,0.1, 0.6,0.2), (u_4, 0.6, 0.2,0.1), (u_5, 0.3, 0.4,0.2), (u_6, 0.5, 0.3,0.1)}
$F_A(e_4)$={(u_1, 0.5, 0.2,0.2), (u_2,0.4, 0.5,0.1), (u_3, 0.5, 0.2,0.2), (u_4, 0.4, 0.2,0.3), (u_5, 0.4, 0.4,0.2), (u_6, 0.5, 0.3,0.1) }

The tabular representation of NSS (F_A,E) is

U	Costly(e_1)	Medium(e_2)	Very cheap(e_4)
u_1	(0.3, 0.4 ,0.2)	(0.4, 0.5,0.1)	(0.5, 0.2,0.2)
u_2	(0.5, 0.4, 0.1)	0.4, 0.2,0.3)	(0.4, 0.5,0.1)
u_3	(0.4, 0.4, 0.2)	(0.1, 0.6,0.2)	(0.5, 0.2,0.2)
u_4	(0.5, 0.2,0.1)	(0.6, 0.2,0.1)	(0.4, 0.2,0.3)
u_5	(0.6, 0.2,0.2)	(0.3, 0.4,0.2)	(0.4, 0.4,0.2)
u_6	(0.5,0.2, 0.2)	(0.5, 0.3,0.1)	(0.5, 0.3,0.1)

Definition 7: Neutrosophic Soft Matrix (Deli. *et. al*, 2015)

Let (F_A,E) is a neutrosophic soft set over U where F_A is a mapping given by $F_A : E \to P(U)$ where P(U) is the collection of all neutrosophic subsets of U. Then the subset of UXE is uniquely defined by $R_A = \{(u,e) : e \in A, u \in F_A(e)\}$ which is called a relation form of (F_A,E) .Now the relation R_A characterized by truth membership function $T_A : U \times E \to [0,1]$, interminancy membership function $I_A : U \times E \to [0,1]$ and falsity membership function $F_A : U \times E \to [0,1]$ where T_A(u,e) is the truth membership value, I_A(u,e) is the interminancy membership value and F_A(u,e) is the falsity membership value of the object u associated with the parameter e.

Let $U = \{u_1, u_2, u_3, ...u_m\}$ be the universe set and $E = \{x_1, x_2, x_3, ...x_n\}$ be the set of parameters. Then R_A can be represented by tabular form as follows:

R_N	e_1	e_2	……	e_n
u_1	$(T_{A_{11}}, I_{A_{11}}, F_{A_{11}})$	$(T_{A_{12}}, I_{A_{12}}, F_{A_{12}})$	……	$(T_{A_{1n}}, I_{A_{1n}}, F_{A_{1n}})$
u_2	$(T_{A_{21}}, I_{A_{21}}, F_{A_{21}})$	$(T_{A_{22}}, I_{A_{22}}, F_{A_{22}})$	……	$(T_{A_{2n}}, I_{2n}, F_{A_{2n}})$
:			……	
u_m	$(T_{A_{m2}}, I_{A_{m2}}, F_{A_{m2}})$	$(T_{A_{m2}}, I_{A_{m2}}, F_{A_{m2}})$	…	$(T_{A_{mn}}, I_{A_{mn}}, F_{A_{mn}})$

Where

$(T_{A_{mn}}, I_{A_{mn}}, F_{A_{mn}})$=($T_A$($u_m$,$e_n$), I_A(u_m,e_n), F_A(u_m,e_n)). If a_{ij}=(T_A(u_i,e_j), I_A(u_i,e_j), F_A(u_i,e_j)) we can define a matrix

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

This is called neutrosophic soft matrix of order mxn corresponding to the neutrosophioc soft set (F_A,E) over U.

Example: Let U={u_1, u_2, u_3, u_4, u_5,u_6}is a universal set. and E={ e_1, e_2, e_3, e_4} be the set of parameters and let
E={ e_1, e_2, e_3}

$F_A(e_1)$={(u_1, 0.3, 0.4,0.2), (u_2, 0.5, 0.4,0.3),(u_3,0.4, 0.5,0.4), (u_4, 0.6, 0.3,0.2), (u_5, 0.8, 0.1,0.3), (u_6, 0.7, 0.2,0.1)}
$F_A(e_2)$={(u_1, 0.4, 0.5,0.2), (u_2, 0.6, 0.2,0.3),(u_3,1, 0,0.4), (u_4, 0.6, 0.2,0.5), (u_5, 0.3, 0.4,0.3), (u_6, 0.5, 0.4,0.4)}
$F_A(e_3)$={(u_1, 0.6, 0.2,0.3), (u_2,0.4, 0.3,0.3), (u_3, 0.5, 0.1,0.4), (u_4, 0.4, 0.2,0.3), (u_5, 0.6, 0.4,0.2), (u_6, 0.7, 0.3,0.2) }

Then the NSS (F_A, E) is a parameterized family {$F_A(e_1)$, $F_A(e_2)$, $F_A(e_3)$ }of all NSS over U and gives an approximate description of the object.
Hence neutrosophic soft matrix can be represented by

$$A = \begin{bmatrix} (0.3,0.4,0.2) & (0.4,0.5,0.2) & (0.6,0.2,0.3) \\ (0.5,0.4,0.3) & (0.6,0.2,0.3) & (0.4,0.3,0.3) \\ (0.4,0.5,0.4) & (0.1,0.0,0.4) & (0.5,0.1,0.4) \\ (0.6,0.3,0.2) & (0.6,0.2,0.5) & (0.4,0.2,0.3) \\ (0.8,0.1,0.3) & (0.3,0.4,0.3) & (0.6,0.4,0.2) \\ (0.7,0.2,0.1) & (0.5,0.4,0.4) & (0.7,0.3,0.2) \end{bmatrix}$$

3. **Operations on Neutrosophic Soft matrices**

3.1 Addition of neutrosophic soft matrices

Let $A=[(T_{A_{ij}},I_{A_{ij}},F_{A_{ij}})]$, $B=[(T_{B_{ij}},I_{B_{ij}},F_{B_{ij}})]$ be two neutrosophic soft matrices. Then the max-min product of the two neutrosophic soft matricesA and B is denoted as $A+B$ is defined as $A+B=[\max(T_{A_{ij}},T_{B_{ij}}),\min(I_{A_{ij}},I_{B_{ij}}),\min(F_{A_{ij}},F_{B_{ij}})]$ for all i and j.

3.2 Max-min product of neutrosophic soft matrices

Let $A=[(T_{A_{ij}},I_{A_{ij}},F_{A_{ij}})]$, $B=[(T_{B_{ij}},I_{B_{ij}},F_{B_{ij}})]$ be two neutrosophic soft matrices. Then the max-min product of the two neutrosophic soft matricesA and B is denoted as $A*B$ is defined as $A*B=[\max\min(T_{A_{ij}},T_{B_{ij}}),\min\max(I_{A_{ij}},I_{B_{ij}}),\min\max(F_{A_{ij}},F_{B_{ij}})]$ for all i and j.

3.3 Transpose of neutrosophic soft matrices

Let $A=[(T_{A_{ij}},I_{A_{ij}},F_{A_{ij}})]$ be a neutrosophic soft matrix. Then the transpose of this neutrosophic soft matrix will be defined by denoted by A^T and is defined by $A^T=[(T_{A_{ji}},I_{A_{ji}},F_{A_{ji}})]$

4. Neutrosophic soft block matrices

In this section, neutrosophic soft block matrices and its related properties will be discussed.

4.1 Neutrosofic soft block matrix

A matrix may be subdivided into sub-matrices by drawing lines parallel to its rows and columns. These sub-matrices may be considered as the elements of the original matrices.

For example

$$A=\begin{bmatrix} (T_{A_{11}},I_{A_{11}},F_{A_{11}}) & (T_{A_{12}},I_{A_{12}},F_{A_{12}}) & : & (T_{A_{13}},I_{A_{13}},F_{A_{13}}) & (T_{A_{14}},I_{A_{14}},F_{A_{14}}) \\ \cdots & \cdots & & \cdots & \cdots \\ (T_{A_{21}},I_{A_{21}},F_{A_{21}}) & (T_{A_{22}},I_{A_{22}},F_{A_{22}}) & : & (T_{A_{23}},I_{A_{23}},F_{A_{23}}) & (T_{A_{24}},I_{A_{24}},F_{A_{24}}) \\ (T_{A_{31}},I_{A_{31}},F_{A_{31}}) & (T_{A_{32}},I_{A_{32}},F_{A_{32}}) & : & (T_{A_{33}},I_{A_{33}},F_{A_{33}}) & (T_{A_{34}},I_{A_{34}},F_{A_{34}}) \end{bmatrix}$$

We may write the above matrix as

$$A=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Where

$$P_{11}=\left[(T_{A_{11}},I_{A_{11}},F_{A_{11}}) \quad (T_{A_{12}},I_{A_{12}},F_{A_{12}})\right]$$

$$P_{12}=\left[(T_{A_{13}},I_{A_{13}},F_{A_{13}}) \quad (T_{A_{14}},I_{A_{14}},F_{A_{14}})\right]$$

$$P_{21}=\begin{bmatrix} (T_{A_{21}},I_{A_{21}},F_{A_{21}}) & (T_{A_{22}},I_{A_{22}},F_{A_{22}}) \\ (T_{A_{31}},I_{A_{31}},F_{A_{31}}) & (T_{A_{32}},I_{A_{32}},F_{A_{32}}) \end{bmatrix}, \quad P_{22}=\begin{bmatrix} (T_{A_{23}},I_{A_{23}},F_{A_{23}}) & (T_{A_{24}},I_{A_{24}},F_{A_{24}}) \\ (T_{A_{33}},I_{A_{33}},F_{A_{33}}) & (T_{A_{34}},I_{A_{34}},F_{A_{34}}) \end{bmatrix}$$

Then

$A=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ is an example of neutrosophic soft block matrix.

So the matrix A is partitioned. The dotted lines divided the matrix into sub-matrices P_{11}, P_{12}, P_{21}, P_{22} are the sub matrices. The matrix A can be partitioned in several ways.

4.2 Square neutrosophic soft block matrix

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If the number of rows and the number of columns of blocks are equal then the matrix is said to be square fuzzy block matrix.

For example

$$A=\begin{bmatrix} (T_{A_{11}},I_{A_{11}},F_{A_{11}}) & (T_{A_{12}},I_{A_{12}},F_{A_{12}}) & : & (T_{A_{13}},I_{A_{13}},F_{A_{13}}) & (T_{A_{14}},I_{A_{14}},F_{A_{14}}) & : & (T_{A_{15}},I_{A_{15}},F_{A_{15}}) & (T_{A_{16}},I_{A_{16}},F_{A_{16}}) \\ (T^A_{21},I^A_{21},F^A_{21}) & (T^A_{22},I^A_{22},F^A_{22}) & : & (T^A_{23},I^A_{23},F^A_{23}) & (T^A_{24},I^A_{24},F^A_{24}) & : & (T_{A_{25}},I_{A_{25}},F_{A_{25}}) & (T_{A_{26}},I_{A_{26}},F_{A_{26}}) \\ \cdots & \cdots & .. & \cdots & \cdots & \cdots & \cdots & \cdots \\ (T_{A_{31}},I_{A_{31}},F_{A_{31}}) & (T_{A_{32}},I_{A_{32}},F_{A_{32}}) & : & (T_{A_{33}},I_{A_{33}},F_{A_{33}}) & (T_{A_{34}},I_{A_{34}},F_{A_{34}}) & : & (T_{A_{35}},I_{A_{35}},F_{A_{35}}) & (T_{A_{36}},I_{A_{36}},F_{A_{36}}) \\ (T_{A_{41}},I_{A_{41}},F_{A_{41}}) & (T_{A_{42}},I_{A_{42}},F_{A_{42}}) & : & (T_{A_{43}},I_{A_{43}},F_{A_{43}}) & (T_{A_{44}},I_{A_{44}},F_{A_{44}}) & : & (T^A_{45},I^A_{45},F^A_{45}) & (T^A_{46},I^A_{46},F^A_{46}) \end{bmatrix}$$

or

$$A=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

is a square fuzzy block matrix since all A_{ij} 's are square blocks.

Numerical Example:

$$A=\begin{bmatrix} (0.5,0.4,0.3) & (0.6,0.7,0.2) & : & (0.4,0.6,0.5) & (0.5,0.4,0.5) & : & (0.5,0.4,0.3) & (0.6,0.4,0.4) \\ (0.4,0.5,0.5) & (0.7,0.2,0.3) & : & (0.8,0.3,0.3) & (0.5,0.4,0.3) & : & (0.8,0.2,0.1) & (0.6,0.2,0.3) \\ \cdots & \cdots & .. & \cdots & \cdots & \cdots & \cdots & \cdots \\ (0.6,0.4,0.3) & (0.7,0.2,0.4) & : & (0.5,0.1,0.3) & (0.6,0.3,0.3) & : & (0.4,0.5,0.4) & (0.7,0.2,0.4) \\ (0.5,0.4,0.5) & (0.6,0.4,0.5) & : & (0.5,0.5,0.3) & (0.6,0.4,0.3) & : & (0.5,0.4,0.5) & (0.6,0.5,0.4) \end{bmatrix}$$

Thus A is an example of square fuzzy block matrix since all the blocks considered here are themselves square neutrosophic soft matrices.

4.3 Rectangular neutrosophic soft block matrix

If the number of rows and the number of columns of blocks are unequal then the matrix is said to be rectangular neutrosophic soft block matrix.

Numerical Example:

$$A=\begin{bmatrix} \\ (0.8,0.3,0.2) & (0.6,0.3,0.3) & : & (0.7,0.3,0.4) & (0.6,0.5,0.3) \\ \cdots\ \cdots & & & \cdots & \cdots \\ (0.5,0.3,0.4) & (0.6,0.4,0.4) & : & (0.5,0.1,0.4) & (0.8,0.3,0.1) \\ (0.6,0.2,0.4) & (0.5,0.3,0.2) & : & (0.5,0.3,0.5) & (0.6,0.3,0.2) \end{bmatrix}$$

The above neutrosophic block matrix is rectangular neutrosophic soft block matrix because each block is not of the same order.

5. Operations on neutrosophic soft block matrices

5.1 Addition of two neutrosophic soft block matrices

Let $A=\begin{bmatrix} A_{11} & : & A_{12} \\ ..... & : & ..... \\ A_{21} & : & A_{22} \end{bmatrix}$ and $B=\begin{bmatrix} B_{11} & : & B_{12} \\ ..... & : & ..... \\ B_{21} & : & B_{22} \end{bmatrix}$ be two matrices of the same order and are partitioned identically, then the addition of two neutrosophic soft block matrices can be defined as

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$$A+B=\begin{bmatrix} A_{11}+B_{11} & : & A_{12}+B_{12} \\ ..... & : & ..... \\ A_{21}+B_{21} & : & A_{22}+B_{22} \end{bmatrix}$$

5.2 Properties of addition of fuzzy block matrices

If three neutrosophic soft block matrices are represented as

$$A=\begin{bmatrix} A_{11} & : & A_{12} \\ ..... & : & ..... \\ A_{21} & : & A_{22} \end{bmatrix}, \ B=\begin{bmatrix} B_{11} & : & B_{12} \\ ..... & : & ..... \\ B_{21} & : & B_{22} \end{bmatrix} \text{ and } C=\begin{bmatrix} C_{11} & : & C_{12} \\ ..... & : & ..... \\ C_{21} & : & C_{22} \end{bmatrix}$$

Then

i. A+B=B+A

ii. A+(B+C)=(A+B+C)

The addition of the above two block matrices will be as

$$A+B=\begin{bmatrix} A_{11}+B_{11} & : & A_{12}+B_{12} \\ ..... & : & ..... \\ A_{21}+B_{21} & : & A_{22}+B_{22} \end{bmatrix}$$

$$B+A=\begin{bmatrix} B_{11}+A_{11} & : & B_{12}+A_{12} \\ ..... & : & ..... \\ B_{21}+A_{21} & : & B_{22}+A_{22} \end{bmatrix}$$

From the above it can be concluded that A+B=B+A.

Similarly it can be proved that A+(B+C)=(A+B)+C

5.3 Max-min operations on neutrosophic soft block matrix

If A and B be two neutrosophic soft block matrices are represented as

$$A=\begin{bmatrix} A_{11} & : & A_{12} \\ ..... & : & ..... \\ A_{21} & : & A_{22} \end{bmatrix}, \ B=\begin{bmatrix} B_{11} & : & B_{12} \\ ..... & : & ..... \\ B_{21} & : & B_{22} \end{bmatrix}$$

Then the product of two neutrosophic soft block matrices will be represented by

$$AB=\begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & : & A_{11}B_{12}+A_{12}B_{22} \\ ..... & : & ..... \\ A_{21}B_{11}+A_{22}B_{21} & : & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

Where each block is conformable for multiplication i.e the number of columns of one block should be equal to the number of rows of the other block which are taken into consideration.

5.4 Transpose of neutrosophic soft block matrix

If $A=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$

be a neutrosophic soft block matrix, then the transpose of that neutrosophic soft block matrix is defined as

$$A^T = \begin{bmatrix} P_{11}{}^T & P_{12}{}^T \\ P_{21}{}^T & P_{22}{}^T \end{bmatrix}$$

Where

$$P_{11}{}^T = \begin{bmatrix} (T_{11}^A, I_{11}^A, F_{11}^A) \\ (T_{12}^A, I_{12}^A, F_{12}^A) \end{bmatrix},$$

$$P_{12}{}^T = \begin{bmatrix} (T_{13}^A, I_{13}^A, F_{13}^A) \\ (T_{14}^A, I_{14}^A, F_{14}^A) \end{bmatrix}$$

$$P_{21}{}^T = \begin{bmatrix} (T_{21}^A, I_{21}^A, F_{21}^A) & (T_{31}^A, I_{31}^A, F_{31}^A) \\ (T_{22}^A, I_{22}^A, F_{22}^A) & (T_{32}^A, I_{32}^A, F_{32}^A) \end{bmatrix}$$

And

$$P_{22}{}^T = \begin{bmatrix} (T_{23}^A, I_{23}^A, F_{23}^A) & (T_{33}^A, I_{33}^A, F_{33}^A) \\ (T_{24}^A, I_{24}^A, F_{24}^A) & (T_{34}^A, I_{34}^A, F_{34}^A) \end{bmatrix}$$

5.5 Properties of transpose of neutrosophic soft block matrix

If A and B be two fuzzy block matrices then the following properties hold:

i. $(A+B)^T = A^T + B^T$

ii. $(A^T)^T = A$

iii. $(kA)^T = kA^T$

Proof of (i)

Let us consider two neutrosophic soft partition matrices as

$$A = \begin{bmatrix} (T^A{}_{11}, I^A{}_{11}, F_{11}{}^A) & (T^A{}_{12}, I^A{}_{12}, F_{12}{}^A) & : & (T^A{}_{13}, I^A{}_{13}, F_{13}{}^A) & (T^A{}_{14}, I^A{}_{14}, F_{14}{}^A) \\ \dots & \dots & & \dots & \dots \\ (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) & (T^A{}_{22}, I^A{}_{22}, F_{22}{}^A) & : & (T^A{}_{23}, I^A{}_{23}, F_{23}{}^A) & (T^A{}_{24}, I^A{}_{24}, F_{24}{}^A) \\ (T^A{}_{31}, I^A{}_{31}, F_{31}{}^A) & (T^A{}_{32}, I^A{}_{32}, F_{32}{}^A) & : & (T^A{}_{33}, I^A{}_{33}, F_{33}{}^A) & (T^A{}_{34}, I^A{}_{34}, F_{34}{}^A) \end{bmatrix}$$

and if it is denoted by

$$A = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \text{ then } A^T = \begin{bmatrix} P_{11}{}^T & P_{12}{}^T \\ P_{21}{}^T & P_{22}{}^T \end{bmatrix} \text{ and so } kA^T = \begin{bmatrix} kP_{11}{}^T & kP_{12}{}^T \\ kP_{21}{}^T & kP_{22}{}^T \end{bmatrix}$$

Again

$$kA = \begin{bmatrix} kP_{11} & kP_{12} \\ kP_{21} & kP_{22} \end{bmatrix} \text{ and so } (kA)^T = \begin{bmatrix} kP^T{}_{11} & kP_{12}^T \\ kP_{21}^T & kP_{22}^T \end{bmatrix}$$

Hence the proposition $(kA)^T = kA^T$

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$(A^T)^T = \begin{bmatrix} (P_{11}^{\ T})^T & (P_{12}^{\ T})^T \\ (P_{21}^{\ T})^T & (P_{22}^{\ T})^T \end{bmatrix}$ and if we find the transposes of the elements which sub-matrices $(P_{11}^T)^T$, $(P_{12}^T)^T$, $(P_{21}^T)^T$, $(P_{22}^T)^T$ of this partitioned neutrosophic matrix then it can easily seen that

$(P_{11}^T)^T = P_{11}, (P_{12}^T)^T = P_{12}, (P_{21}^T)^T = P_{21}$ and $(P_{22}^T)^T = P_{22}$

Hence the proposition $(A^T)^T = A$ holds true.

$$B = \begin{bmatrix} (T^B{}_{11}, I^B{}_{11}, F_{11}{}^B) & (T^B{}_{12}, I^B{}_{12}, F_{12}{}^B) & : & (T^B{}_{13}, I^B{}_{13}, F_{13}{}^B) & (T^B{}_{14}, I^B{}_{14}, F_{14}{}^B) \\ \dots & \dots & & \dots & \dots \\ (T^B{}_{21}, I^B{}_{21}, F_{21}{}^B) & (T^B{}_{22}, I^B{}_{22}, F_{22}{}^B) & : & (T^B{}_{23}, I^B{}_{23}, F_{23}{}^B) & (T^B{}_{24}, I^B{}_{24}, F_{24}{}^B) \\ (T^B{}_{31}, I^B{}_{31}, F_{31}{}^B) & (T^B{}_{32}, I^B{}_{32}, F_{32}{}^B) & : & (T^B{}_{33}, I^B{}_{33}, F_{33}{}^B) & (T^B{}_{34}, I^B{}_{34}, F_{34}{}^B) \end{bmatrix}$$

$$B = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

$A + B = \begin{bmatrix} P_{11} + Q_{11} & : & P_{12} + Q_{12} \\ \dots\dots & : & \dots\dots \\ P_{21} + Q_{21} & : & P_{22} + Q_{22} \end{bmatrix}$ and so $(A+B)^T = \begin{bmatrix} (P_{11} + Q_{11})^T & : & (P_{12} + Q_{12})^T \\ \dots\dots & : & \dots\dots \\ (P_{21} + Q_{21})^T & : & (P_{22} + Q_{22})^T \end{bmatrix}$

$P_{11} = \left[(T^A{}_{11}, I^A{}_{11}, F_{11}^A) \quad (T^A{}_{12}, I^A{}_{12}, F_{12}^A) \right]$ and $Q_{11} = \left[(T^B{}_{11}, I^B{}_{11}, F_{11}^B) \quad (T^B{}_{12}, I^B{}_{12}, F_{12}^B) \right]$

$$P_{11} + Q_{11} = [\{\max(T^A{}_{11}, T^B{}_{11}), \min(I^A{}_{11}, I^B{}_{11}), \min(F_{11}^A, F^B{}_{11})\},$$
$$\{\max(T^A{}_{12}, T^B{}_{12}), \min(I^A{}_{12}, I^B{}_{12}), \min(F_{12}^A, F^B{}_{12})\}]$$

$$(P_{11} + Q_{11})^T = \begin{bmatrix} \{\max(T^A{}_{11}, T^B{}_{11}), \min(I^A{}_{11}, I^B{}_{11}), \min(F_{11}^A, F^B{}_{11})\} \\ \{\max(T^A{}_{12}, T^B{}_{12}), \min(I^A{}_{12}, I^B{}_{12}), \min(F_{12}^A, F^B{}_{12})\} \end{bmatrix}$$

$P_{11}^T = \begin{bmatrix} (T^A{}_{11}, I^A{}_{11}, F_{11}^A) \\ (T^A{}_{12}, I^A{}_{12}, F_{12}^A) \end{bmatrix}$ and $Q_{11}^T = \begin{bmatrix} (T^B{}_{11}, I^B{}_{11}, F_{11}^B) \\ (T^B{}_{12}, I^B{}_{12}, F_{12}^B) \end{bmatrix}$

$$(P_{11})^T + (Q_{11})^T = \begin{bmatrix} \{\max(T^A{}_{11}, T^B{}_{11}), \min(I^A{}_{11}, I^B{}_{11}), \min(F_{11}^A, F^B{}_{11})\} \\ \{\max(T^A{}_{12}, T^B{}_{12}), \min(I^A{}_{12}, I^B{}_{12}), \min(F_{12}^A, F^B{}_{12})\} \end{bmatrix}$$

Hence it can be seen that $(P_{11} + Q_{11})^T = P_{11}{}^T + Q_{11}{}^T$

Similarly for the others.

Then we have

$A^T = \begin{bmatrix} P_{11}{}^T & P_{12}{}^T \\ P_{21}{}^T & P_{22}{}^T \end{bmatrix}$ and $B^T = \begin{bmatrix} Q_{11}^T & Q_{12}{}^T \\ Q_{21}{}^T & Q_{22}{}^T \end{bmatrix}$

And so

$$A^T + B^T = \begin{bmatrix} P_{11}{}^T + Q_{11}{}^T & P_{12}{}^T + Q_{12}{}^T \\ P_{21}{}^T + Q_{21}{}^T & P_{22}{}^T + Q_{22}{}^T \end{bmatrix}$$

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6. Conclusions

In this article, the concept of neutrosophic soft block matrices are developed and accordingly various types of neutrosophic soft block matrices are studied. Some operations on neutrosophic soft block matrices are also discussed. Thereafter some properties of neutrosophic soft block matrices are studied and it is found that neutrosophic soft block matrices behave in the same way as those of other block matrices that exist in the literature.

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Pairwise Pythagorean Neutrosophic Strongly Irresolvable Spaces (with dependent neutrosophic components between T and F)

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Abstract

In this paper, we have define the pairwise Pythagorean Neutrosophic (for short,pairwise PN) bitopological spaces (with dependent neutrosophic components between T and F). We also study the Pairwise PN strongly irresolvable spaces .The conditions under which pairwise PN strongly irresolvable spaces become pairwise PN first category spaces and pairwise PN Baire spaces are also investigated.
Keywords: PN bitopology,PN strongly irresolvablen spaces,PN Baire.

1 Introduction

Fuzzy sets were introduced by Zadeh [18] and he discussed only membership function. The fuzzy topology concept was first introduced by C.L.Chang [6] in 1968.After the extensions of fuzzy set theory Atanassov [4] generalized this concept and introduced a new concept called intuitionistic fuzzy set (IFS). Yager [16] familiarized the model of Pythagorean fuzzy set.

IFS was failed to deal with indeterminate and inconsistent information which exist in beliefs system, therefore, Smarandache [13] in 1995 introduced new concept known as neutrosophic set(NS) which generalizes fuzzy sets and intuitionistic fuzzy sets and so on. A neutrosophic set includes truth membership, falsity membership and indeterminacy membership. In 2006, F.Smarandache introduced, for the first time, the degree of dependence (and consequently the degree of independence) between the components of the fuzzy set, and also between the components of the neutrosophic set. In 2016, the refined neutrosophic set was generalized to the degree of dependence or independence of subcomponents [14]. A.Kandil [9] introduced the concept of fuzzy bitopological spaces as a generalization of fuzzy topological spaces.Thagaraj and Balasubramanian [15] introduced the concept of fuzzy resolvable and irresolvable spaces.Jansi,Mohana and Florentin Smarandache [8] were firstly studied the concept of Pythagorean Neutrosophic sets with T and F as dependent neutrosophic components.

In this paper we study the pairwise PN strongly irresolvable spaces .Also we studied the conditions under which pairwise PN bitopological strongly irresolvable spaces spaces become pairwise PN frirst category spaces and pairwise PN Baire spaces are investigated.

2 Preliminaries

Definition 2.1 (16)**. (Pythagorean Fuzzy Set)** Let X be a non-empty set and I the unit interval $[0,1]$. A PF set P is an object having the form $P=\{(x,\mu_P(x),v_P(x)):x\in X\}$ where the function $\mu_P:X\to[0,1]$ and $v_P:X\to[0,1]$ denote respectively the degree of membership and degree of non-membership of each element $x\in X$ to the set P, and $0\leq(\mu_P(x))^2+(v_P(x))^2\leq 1$ for each $x\in X$.

Definition 2.2 (11)**.** Let X be a non-empty set (universe). A neutrosophic set A on X is an object of the form: $A=\{(x,T_A(x),I_A(x),F_A(x)):x\in X\}$,Where $T_A(x),I_A(x),F_A(x)\in[0,1]$,$0\leq T_A(x)+I_A(x)+F_A(x)\leq 2$,for all x in X. $T_A(x)$ is the degree of membership, $I_A(x)$ is the degree of inderminancy and $F_A(x)$ is the degree of non-membership. Here $T_A(x)$ and $F_A(x)$ are dependent components and $I_A(x)$ is an independent components.

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Definition 2.3 (6). **(Pythagorean neutrosophic sets [PN-sets](with T and F are dependent neutrosophic components)** Let X be a non-empty set. A Pythagorean neutrosophic sets with T and F are dependent neutrosophic components (PN) $A = \{(X, T_A(x), I_A(x), F_A(x) : x \in X\}$ where $T_A : X \to [0,1], I_A : X \to [0,1]$ and $F_A : X \to [0,1]$ are the mappings such that $0 \leq T_A^2(x) + I_A^2(x) + F_A^2(x) \leq 2$ and $T_A(x)$ denote the membership degree,$I_A(x)$ denote the Indeterminacy and $F_A(x)$ denote the non-membership degree.Here T and F are dependent neutrosophic components and I is an independent components.

Definition 2.4 (6). Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be two PNs,then their operations are defined as follows:
(1)$A \subseteq B$ if and if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x)$
(2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
(3) $A \cup B = \{(x, max(T_A, T_\mu), min(I_A, I_\mu), min(F_A, T_\mu) : x \in X\}$
(4)$A \cap \mu = \{(x, min(T_A, T_B), max(I_A, I_B), max(F_A, T_B) : x \in X\}$
(5)$A^C = \{(x, F_A, I_A, F_A) : x \in X\}$.

3 PN Bitopological Spaces

Definition 3.1. A Pythagorean neutrosophic (with T and F are dependent neutrosophic components) topology (PNT in Short) on X is a family ρ of PN-sets in X satisfying the following axioms:

(1) $0_X, 1_X \in \rho$

(2)$G_1 \cap G_2 \in \rho$, for any $G_1, G_2 \in \rho$

(3) $\cup G_i \in \rho$ for any family $\{G_i / i \in J\} \subseteq \rho$.Note that $0_X = (0,1,1)$ and $1_X = (1,0,0)$.

In this case the pair (X, ρ) is called a Pythagorean neutrosophic sets with T and F are dependent neutrosophic components topological space (PNTS in Short) and any PNTS in ρ is known as a Pythagorean neutrosophic sets with T and F are dependent neutrosophic components open set (PNOS in Short) in X.

The Complement A^c of a PNOS A in a PNTS (X, ρ) is called a Pythagorean neutrosophic sets with T and F are dependent neutrosophic components closed set (PNCS in Short) in X.

Definition 3.2. Let (X, ρ) be a PNTS and be a PN in X. Then the PN interior and closure of a PN closure are defined by

$$PNint(A) = \cup \{G / GisaPNOSinXandG \subseteq A\}$$

$$PNcl(A) = \cap \{K / KisaPNCSinXandA \subseteq K\}.$$

Note that for any PN A in (X, ρ), we have $(PNcl(A))^c = PNint(A^c)$ and $(PNint(A))^c = PNcl(A^c)$.Also,note that A is a PN closed set iff $PNcl(A) = A$ and A is a PN open set iff $PNint(A) = A$.

Definition 3.3. A set X on which are defined two (arbitrary)PN topologies ρ_1 and ρ_2) is called PN bitopological spaces and denoted by (X, ρ_1, ρ_2).
We shall write $PNint_{\rho_i}(A)$ and $PNcl_{\rho_i}(A)$ to mean respectively the PN interior and PN closure of PN set A with respect to the ρ_i in (X, ρ_1, ρ_2).

Definition 3.4. A PN-set A in a PN bitopological space (X, ρ_1, ρ_2) is called a pairwise PN open set if $A \in \rho_i (i = 1, 2)$.The complement of pairwise PN open set in (X, ρ_1, ρ_2) is called a pairwise PN closed set.

Definition 3.5. A PN-set A in a PN bitopological space (X, ρ_1, ρ_2) is called pairwise PN semi open set if $A \subseteq PNint_{\rho_i} PNcl_{\rho_i}(A)(i = 1, 2)$ and PN semi closed set if $PNint_{\rho_i}(PNcl_{\rho_i}(A)) \subseteq A(i = 1, 2)$.

Definition 3.6. A PN-set A in a PN bitopological space (X, ρ_1, ρ_2) is called a pairwise PN dense set if $PNcl_{\rho_1} PNcl_{\rho_2}(A) = PNcl_{\rho_2} PNcl_{\rho_1}(A) = 1_X$ in (X, ρ_1, ρ_2).

Definition 3.7. A PN-set A in a PN bitopological space (X, ρ_1, ρ_2) is called pairwise PN nowhere dense if $PNint_{\rho_1} PNcl_{\rho_2}(A) = PNint_{\rho_2} PNcl_{\rho_1}(A) = 0_X$
in (X, ρ_1, ρ_2).

Definition 3.8. Let (X, ρ_1, ρ_2) be a PN bitopological space.A PN-set A in (X, ρ_1, ρ_2) is called a pairwise PN first category set if $A = \bigcup_{i=1}^{\infty}(A_i)$, where (A_i)'s are pairwise PN nowhere dense sets in (X, ρ_1, ρ_2).Any other PN set in (X, ρ_1, ρ_2) is said to be a pairwise PN second category set in (X, ρ_1, ρ_2).

Definition 3.9. If A is a pairwise PN first category set in a PN bitopological space (X, ρ_1, ρ_2), then the PN A^c is called a pairwise PN residual set in (X, ρ_1, ρ_2).

Definition 3.10. A PN-set A in a PN bitopological space (X, ρ_1, ρ_2) is called pairwise PN F_σ-set in (X, ρ_1, ρ_2) if $A = \bigcup_{i=1}^{\infty}(A_i)$ where $(A_i)^c \in \rho_i$.

Definition 3.11. A PN-set A in a PN bitopological space (X, ρ_1, ρ_2) is called pairwise PN G_δ-set in (X, ρ_1, ρ_2) if $A = \bigcap_{i=1}^{\infty}(A_i)$ where $A_i \in \rho_i$.

Definition 3.12. A PN bitopological space (X, ρ_1, ρ_2) is called pairwise PN first category space if the PN set 1_X is a pairwise PN first category set in (X, ρ_1, ρ_2).That is, $1_X = \bigcup_{i=1}^{\infty}(A_i)$, where A_i's are pairwise PN nowhere dense sets in (X, ρ_1, ρ_2).
Otherwise (X, ρ_1, ρ_2) will be called a pairwise PN second category space.

Theorem 3.13. Let (X, ρ_1, ρ_2) be a pairwise PN strongly irresolvable space.Then, A is a pairwise PN dense set in (X, ρ_1, ρ_2), if and only if (A^c) is a pairwise PN nowhere dense set in (X, ρ_1, ρ_2).

Proof. Let A be a pairwise PN dense set in (X, ρ_1, ρ_2).Since (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable space $PNcl_{\rho_1}PNint_{\rho_2}(A) = 1_X = PNcl_{\rho_2}PNint_{\rho_1}(A)$.
This implies that $(PNcl_{\rho_1}PNint_{\rho_2}(A))^c = 0_X = (PNcl_{\rho_2}PNint_{\rho_1}(A))^c$.
Therefore $PNint_{\rho_1}PNcl_{\rho_2}(A^c) = 0_X = PNint_{\rho_2}PNcl_{\rho_1}(A^c)$ and hence A^c is a pairwise PN nowhere dense set. □

Theorem 3.14. Let (X, ρ_1, ρ_2) be a pairwise PN dense and pairwise PN G_δ-set in a PN bitopological space (X, ρ_1, ρ_2), then (A^c) is a pairwise PNs σ-nowhere dense set in (X, ρ_1, ρ_2).

Proof. Let (X, ρ_1, ρ_2) be a pairwise PN dense and pairwise PN G_δ-set in a PN bitopological space (X, ρ_1, ρ_2) .Then $PNcl_{\rho_1}PNcl_{\rho_2}(A) = PNcl_{\rho_2}PNcl_{\rho_1}(A) = 1_X$.
This implies that
$(PNcl_{\rho_1}PNcl_{\rho_2}(A))^c = (PNcl_{\rho_2}PNcl_{\rho_1}(A))^c = 0_X$
and hence we have $PNint_{\rho_1}PNint_{\rho_2}(A^c) = 0_X = PNint_{\rho_2}PNint_{\rho_1}(A^c)$.
Also, since A is a pairwise PN G_δ-set,A^c is a pairwise PN F_σ-set in (X, ρ_1, ρ_2).Hence A^c is a pairwise PN F_σ-set in (X, ρ_1, ρ_2) such that $PNint_{\rho_1}PNint_{\rho_2}(A^c) = 0_X = PNint_{\rho_2}PNint_{\rho_1}(A^c)$.
Thus (A^c) is a pairwise PNs σ-nowhere dense set in (X, ρ_1, ρ_2). □

4 Pairwise PN Strongly Irresolvable Spaces

Definition 4.1. A PN bitopological space (X, ρ_1, ρ_2) is said to be a pairwise PN strongly irresolvable space if $PNcl_{\rho_1}PNint_{\rho_2}(A) = 1_X = PNcl_{\rho_2}PNint_{\rho_1}(A)$, for each pairwise PN dense set A in (X, ρ_1, ρ_2).That is, (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable space if
$PNcl_{\rho_1}PNcl_{\rho_2}(A) = 1_X = PNcl_{\rho_2}PNcl_{\rho_1}(A)$ for a PN-set A in (X, ρ_1, ρ_2), then
$PNcl_{\rho_1}PNint_{\rho_2}(A) = 1_X = PNcl_{\rho_2}PNint_{\rho_1}(A)$ in (X, ρ_1, ρ_2).

Theorem 4.2. Let (X, ρ_1, ρ_2) be a pairwise PN strongly irresolvable space.Then, A is a pairwise PN dense set in (X, ρ_1, ρ_2), if and only if (A^c) is a pairwise PN nowhere dense set in (X, ρ_1, ρ_2).

Proof. Let A be a pairwise PN dense set in (X, ρ_1, ρ_2).
Since (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable space
$PNcl_{\rho_1}PNint_{\rho_2}(A) = 1_X = PNcl_{\rho_2}PNint_{\rho_1}(A)$.
This implies that $(PNcl_{\rho_1}PNint_{\rho_2}(A))^c = 0_X = (PNcl_{\rho_2}PNint_{\rho_1}(A))^c$.
Therefore $PNint_{\rho_1}PNcl_{\rho_2}(A^c) = 0_X = PNint_{\rho_2}PNcl_{\rho_1}(A^c)$ and hence A^c is a pairwise PN nowhere dense set. □

Theorem 4.3. Let (X, ρ_1, ρ_2) be a pairwise PN dense and pairwise PN G_δ-set in a PN bitopological space (X, ρ_1, ρ_2), then (A^c) is a pairwise PNs σ-nowhere dense set in (X, ρ_1, ρ_2).

Proof. Let (X, ρ_1, ρ_2) be a pairwise PN dense and pairwise PN G_δ-set in a PN bitopological space (X, ρ_1, ρ_2) .
Then $PNcl_{\rho_1} PNcl_{\rho_2}(A) = PNcl_{\rho_2} PNcl_{\rho_1}(A) = 1_X$.
This implies that
$(PNcl_{\rho_1} PNcl_{\rho_2}(A))^c = (PNcl_{\rho_2} PNcl_{\rho_1}(A))^c = 0_X$ and
hence we have $PNint_{\rho_1} PNint_{\rho_2}(A^c) = 0_X = PNint_{\rho_2} PNint_{\rho_1}(A^c)$.
Also, since A is a pairwise PN G_δ-set,A^c is a pairwise PN F_σ-set in (X, ρ_1, ρ_2).
Hence A^c is a pairwise PNs F_σ-set in (X, ρ_1, ρ_2) such that
$PNint_{\rho_1} PNint_{\rho_2}(A^c) = 0_X = PNint_{\rho_2} PNint_{\rho_1}(A^c)$.
Thus (A^c) is a pairwise PN σ-nowhere dense set in (X, ρ_1, ρ_2). □

Proposition 4.4. If $PNint_{\rho_i}(B) = 0_X (i = 1, 2)$ for a PN-set B in a pairwise PN strongly irresolvable space (X, ρ_1, ρ_2), then B is a pairwise PN nowhere dense set in (X, ρ_1, ρ_2).

Proof. Let B be a PN-set in (X, ρ_1, ρ_2) such that $PNint_{\rho_i}(B) = 0_X$
$(i = 1, 2)$.
Then, $PNint_{\rho_1} PNint_{\rho_2}(B) = PNint_{\rho_1}(0_X) = 0_X$ and
$PNint_{\rho_2} PNint_{\rho_1}(B) = PNint_{\rho_2}(0_X) = 0_X$ in (X, ρ_1, ρ_2).
Then, we have $(PNint_{\rho_1} PNint_{\rho_2}(B))^c = 1_X$ and
$(PNint_{\rho_2} PNint_{\rho_1}(B^c) = 1_X$ and
hence $PNcl_{\rho_1} PNcl_{\rho_2}(B^c) = 1_X$ and $PNcl_{\rho_2} PNcl_{\rho_1}(B^c) = 1_X$.
That is, B^c is a pairwise PN dense set in (X, ρ_1, ρ_2).
Since (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable space,
$PNcl_{\rho_1} PNint_{\rho_2}(B^c) = 1_X = PNcl_{\rho_2} PNint_{\rho_1}(B^c)$, for the pairwise PN dense set B^c in (X, ρ_1, ρ_2).
Then, we have $(PNcl_{\rho_1} PNint_{\rho_2}(B^c))^c = 0_X$ and
$(PNcl_{\rho_2} PNint_{\rho_1}(B^c))^c = 0_X$ and hence $PNint_{\rho_1} PNcl_{\rho_2}(B) = 0_X$ and
$PNint_{\rho_2} PNcl_{\rho_1}(B) = 0_X$.
Therefore B is a pairwise PN nowhere dense set in (X, ρ_1, ρ_2). □

Proposition 4.5. If (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable space if $PNcl_{\rho_1} PNint_{\rho_2}(A) \neq 1_X$ and $PNcl_{\rho_2} PNint_{\rho_1}(A) \neq 1_X$ for a PN A in (X, ρ_1, ρ_2), then $PNcl_{\rho_1} PNcl_{\rho_2} \neq 1_X$ and $PNcl_{\rho_2} PNcl_{\rho_1}(A) \neq 1_X$ in (X, ρ_1, ρ_2).

Proof. Let $PNcl_{\rho_1} PNint_{\rho_2}(A) \neq 1_X$ and $PNcl_{\rho_2} PNcl_{\rho_1}(A) \neq 1_X$, for a PN-set A in the PN bitopological space (X, ρ_1, ρ_2).
Suppose that $PNcl_{\rho_1} PNcl_{\rho_2}(A) = 1_X$ and
$PNcl_{\rho_2} PNcl_{\rho_1}(A) = 1_X$ in (X, ρ_1, ρ_2).
Since (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable
space,$PNcl_{\rho_1} PNcl_{\rho_2}(A) = 1_X$ and $PNcl_{\rho_2} PNcl_{\rho_1}(A) = 1_X$ in (X, ρ_1, ρ_2), will imply that
$PNcl_{\rho_1} PNint_{\rho_2}(A) = 1_X$ and $PNcl_{\rho_2} PNint_{\rho_1}(A) = 1_X$
in (X, ρ_1, ρ_2),a contradiction to the hypothesis.
Hence we must have $PNcl_{\rho_1} PNcl_{\rho_2}(A) \neq 1_X$ and
$PNcl_{\rho_2} PNcl_{\rho_1}(A) \neq 1_X$ in (X, ρ_1, ρ_2). □

Proposition 4.6. If $PNint_{\rho_1} PNcl_{\rho_2}(A) \neq 0_X$ and $PNint_{\rho_2} PNcl_{\rho_1}$
$\neq 0_X$, for a PN A in a pairwise PN strongly irresolvable space (X, ρ_1, ρ_2), then $PNint_{\rho_1} \neq 0_X$ and $PNint_{\rho_2}(A) \neq 0_X$ in (X, ρ_1, ρ_2).

Proof. Suppose that $PNint_{\rho_1}(A) = 0_X$ and $PNint_{\rho_2}(A) = 0_X$ for a PN-set A in a pairwise PN strongly irresolvable space (X, ρ_1, ρ_2).
Then, $(PNint_{\rho_1} PNint_{\rho_2}(A))^c = (PNint_{\rho_1}(0_X))^c = 1_X$ and
$(PNint_{\rho_2} PNint_{\rho_1}(A))^c = (PNint_{\rho_2}(0_X))^c = 1_X$.
Then $PNcl_{\rho_1} PNcl_{\rho_2}(A^c) = 1_X$ and $PNcl_{\rho_2} PNcl_{\rho_1}(A^c) = 1_X$.
Hence (A^c) is a pairwise PNs dense set in (X, ρ_1, ρ_2).Since (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable space, $PNcl_{\rho_1} PNint_{\rho_2}(A^c) = 1_X$ and $PNcl_{\rho_2} PNint_{\rho_1}(A^c) = 1_X$.
Then, we will have $(PNint_{\rho_1} PNcl_{\rho_2}(A))^c = 1_X$ and
$(PNint_{\rho_2} PNcl_{\rho_1}(A))^c = 1_X$ and
hence $PNint_{\rho_1} PNcl_{\rho_2}(A) = 0_X$ and
$PNint_{\rho_2} PNcl_{\rho_1}(A) = 0_X$, a contradiction.
Hence we must have $PNint_{\rho_1} \neq 0_X$ and $PNint_{\rho_2}(A) \neq 0_X$ in (X, ρ_1, ρ_2). □

Theorem 4.7. If A is a pairwise PN σ-nowhere dense set in a pairwise PN strongly irresolvable space (X,ρ_1,ρ_2), then A is a pairwise PN nowhere dense set and pairwise PN F_σ-set in (X,ρ_1,ρ_2).

Proof. Let A be a pairwise PN σ-nowhere dense set in (X,ρ_1,ρ_2).
Then ¸A is a pairwise PN F_σ-set in (X,ρ_1,ρ_2).
Since $PNint_{\rho_1}PNint_{\rho_2}(A)=0_X=PNint_{\rho_2}PNint_{\rho_1}(A)$.
Then $(PNint_{\rho_1}PNint_{\rho_2}(A))^c = 1_X = (PNint_{\rho_2}PNint_{\rho_1}(A))^c$ implies that $PNcl_{\rho_1}PNcl_{\rho_2}(A^c) = 1_X = PNcl_{\rho_2}PNcl_{\rho_1}(A^c)$.
Hence,(A^c) is a pairwise fuzzy dense set in (X,ρ_1,ρ_2). Since (X,ρ_1,ρ_2) is a pairwise fuzzy strongly irresolvable space, for the pairwise fuzzy dense set (A^c) in (X,ρ_1,ρ_2),we have
$PNcl_{\rho_1}PNint_{\rho_2}(A^c)=1_X=PNcl_{\rho_2}PNint_{\rho_1}(A^c)$.
Then $(PNint_{\rho_1}PNcl_{\rho_2}(A))^c=0_X=(PNint_{\rho_2}PNcl_{\rho_1}(A))^c$ implies that $PNint_{\rho_1}PNcl_{\rho_2}(A)$ and $PNint_{\rho_2}PNcl_{\rho_1}(A)=0_X$.
Therefore A is a pairwise PN nowhere dense set and pairwise PN F_σ-set in (X,ρ_1,ρ_2). □

Proposition 4.8. If A is a pairwise PN σ-nowhere dense set in a pairwise PN strongly irresolvable space (X,ρ_1,ρ_2), then A is a pairwise PN semi closed set in (X,ρ_1,ρ_2).

Proof. Let A be a pairwise PN σ-nowhere dense set in a pairwise PN strongly irresolvable space (X,ρ_1,ρ_2).Then, by theorem 4.7, A is a pairwise PN nowhere dense set in (X,ρ_1,ρ_2) and hence $PNint_{\rho_1}PNcl_{\rho_2}(A)$
$=0_X$ and $PNint_{\rho_2}PNcl_{\rho_1}(A)=0_X$.
Then, we have $PNint_{\rho_1}PNcl_{\rho_2}(A)\subseteq A$ and
$PNint_{\rho_2}PNcl_{\rho_1}(A)\subseteq A$.
Therefore A is a pairwise PN semi-closed set in (X,ρ_1,ρ_2). □

Proposition 4.9. If A is a pairwise PN dense and pairwise PN G_δ-set in a pairwise strongly irresolvable space (X,ρ_1,ρ_2), then A is a pairwise PN semi-open set in (X,ρ_1,ρ_2).

Proof. Let A be a pairwise PN dense and pairwise PN G_δ-set in (X,ρ_1,ρ_2).
Then, by theorem 4.3, (A^c) is a pairwise PN σ-nowhere dense set in (X,ρ_1,ρ_2).
Since (X,ρ_1,ρ_2) is a pairwise PN strongly irresolvable space, by proposition 4.8, (A^c) is a pairwise PN semi-closed set in (X,ρ_1,ρ_2).Hence A is a pairwise PN semi-open set in (X,ρ_1,ρ_2). □

Proposition 4.10. If $A_i\subseteq(A_j)^c(i\neq j)$, where A_i is a pairwise PN dense set in a pairwise PN strongly irresolvable space (X,ρ_1,ρ_2), then A_j is a pairwise PN nowhere dense set in (X,ρ_1,ρ_2).

Proof. Let A_i be a pairwise PN dense set in (X,ρ_1,ρ_2).
Since (X,ρ_1,ρ_2) is a pairwise PN strongly irresolvable space, for the pairwise PN dense set A_i, we have
$PNcl_{\rho_1}PNint_{\rho_2}(A_i)=1_X$ and $PNcl_{\rho_2}PNint_{\rho_1}(A_i)=1_X$.
Now $A_i\subseteq(A_j)^c(i\neq j)$ implies that
$PNcl_{\rho_1}PNint_{\rho_2}(A_i)\subseteq PNcl_{\rho_1}PNint_{\rho_2}((A_j)^c)$ and
$PNcl_{\rho_2}PNint_{\rho_1}(A_i)\subseteq PNcl_{\rho_2}PNint_{\rho_1}(A_j)^c$.
Then, we have $1_X\subseteq PNcl_{\rho_1}PNint_{\rho_2}(A_j)^c$ and
$1_X\subseteq PNcl_{\rho_2}PNint_{\rho_1}(A_j)^c$.
That is, $PNcl_{\rho_1}PNint_{\rho_2}((A_j)^c)=1_X$ and
$PNcl_{\rho_2}PNint_{\rho_1}((A_j)^c)=1_X$.
Hence, $PNint_{\rho_1}PNcl_{\rho_2}(A_j)=0_X$ and $PNint_{\rho_2}PNcl_{\rho_1}(A_j)=0_X$.
Therefore A_j is a pairwise PN nowhere dense set in (X,ρ_1,ρ_2). □

Definition 4.11. A PN bitopological space (X,ρ_1,ρ_2) is called a pairwise PN σ-Baire space
if $PNint_{\rho_i}(\bigcup_{k=1}^{\infty}(A_k))=0_X,(i=1,2)$
where A_k's are pairwise PN σ-nowhere dense sets in (X,ρ_1,ρ_2).

Proposition 4.12. If (X,ρ_1,ρ_2) is a pairwise PN strongly irresolvable space and pairwise PN σ-Baire space, then $PNcl_{\rho_i}(\bigcap_{k=1}^{\infty}(B_k))=1_X(i=1,2)$ where B_k's are pairwise PN semi-open sets in (X,ρ_1,ρ_2).

Proof. Let (X,ρ_1,ρ_2) be a pairwise PN strongly irresolvable space and pairwise PN σ-Baire space.Let A_k's be pairwise PN σ-nowhere dense sets in (X,ρ_1,ρ_2).Since (X,ρ_1,ρ_2) is a pairwise PN σ-Baire space,
$PNint_{\rho_i}(\bigcup_{k=1}^{\infty}(A_k))=0_X,(i=1,2)$ where A_k's are pairwise PN σ-nowhere dense sets in (X,ρ_1,ρ_2).
Then, $(PNint_{\rho_i}(\bigcup_{k=1}^{\infty}(A_k)))^c$
$=1_X$.This implies that $PNcl_{\rho_i}(\bigcap_{k=1}^{\infty}((A_k)^c))=1_X$.By proposition 4.8, the pairwise PN σ-nowhere dense sets A_k's in the pairwise PN strongly irresolvable space (X,ρ_1,ρ_2), are pairwise PN semi-closed sets in

(X,ρ_1,ρ_2) and hence $((A_k)^c)$'s are pairwise PN semi-open sets in (X,ρ_1,ρ_2).Let $B_k = (A_k)^c$.Hence we have $PNcl_{\rho_i}(\bigcap_{k=1}^{\infty}(B_k)) = 1_X, (i = 1, 2)$, where B_k's are pairwise PN semi-open sets in (X,ρ_1,ρ_2). □

Definition 4.13. A PN bitopological space (X,ρ_1,ρ_2) is called a pairwise PN Baire space if
$PNint_{\rho_i}(\bigcup_{k=1}^{\infty}(A_k)) = 0_X, (i = 1, 2)$
where A_k's are pairwise PN nowhere dense sets in (X,ρ_1,ρ_2).

Proposition 4.14. If $A_i \subseteq (A_j)^c (i \neq j)$, where A_i is a pairwise PN dense set in a pairwise PN strongly irresolvable space (X,ρ_1,ρ_2) and if
$PNcl_{PN\rho_k}(\bigcap_{i=1}^{\infty}(A_i)) = 1_X (k = 1, 2)$, then (X,ρ_1,ρ_2) is a pairwise PN Baire space.

Proof. Let $A_i \subseteq (A_j)^c (i \neq j)$, where A_i is a pairwise PN dense set in a pairwise PN strongly irresolvable space (X,ρ_1,ρ_2).Then, by proposition 4.10, A_i's are pairwise PN nowhere dense sets in (X,ρ_1,ρ_2).Now $A_i \subseteq (A_j)^c$, implies that $PNcl_{PN\rho_k}(\bigcap_{i=1}^{\infty}(A_i)) \subseteq PNcl_{PN\rho_k}(\bigcap_{j=1}^{\infty}((A_j)^c))$.Then we have $1_X \subseteq PNcl_{PN\rho_k}(\bigcap_{j=1}^{\infty}((A_j)^c))$ and hence $1_X \subseteq (PNint_{PN\rho_k}(\bigcup_{j=1}^{\infty}(A_j)))^c$.This implies that $PNint_{PN\rho_k}(\bigcup_{j=1}^{\infty}(A_j)) \subseteq 0_X$.That is, $PNint_{PN\rho_k}(\bigcap_{j=1}^{\infty}(A_j)) = 0_X$. Hence $PNint_{PN\rho_k}(\bigcup_{j=1}^{\infty}(A_j)) = 0_X$, where A_j's are pairwise PN nowhere dense sets in (X,ρ_1,ρ_2), implies that (X,ρ_1,ρ_2) is a pairwise PN Baire space. □

Proposition 4.15. If $PNcl_{\rho_i}(\bigcap_{k=1}^{\infty}(A_k) = 1_X (i = 1, 2)$ where A_k's are pairwise PN dense and pairwise PN G_σ-sets in a pairwise PN strongly irresolvable space (X,ρ_1,ρ_2), then (X,ρ_1,ρ_2) is a pairwise PN Baire space.

Proof. Now $PNcl_{\rho_i}(\bigcap_{k=1}^{\infty}(A_k)) = 1_X (i = 1, 2)$,implies that
$PNint_{\rho_i}(\bigcup_{k=1}^{\infty}((A_k)^c)) = 0_X$.Since A_k's are pairwise PN dense and pairwise PN G_σ-sets in a pairwise PN strongly irresolvable space (X,ρ_1,ρ_2), by theorem 4.3 , $((A_k)^c)$'s are pairwise PN σ-nowhere dense sets in (X,ρ_1,ρ_2).Also, by theorem 4.7, $((A_k)^c)$'s are pairwise PN nowhere dense sets in (X,ρ_1,ρ_2).
Hence $PNint_{\rho_i}(\bigcup_{k=1}^{\infty}((A_k)^c)) = 0_X$, where $((A_k)^c)$'s are pairwise PN nowhere dense sets in (X,ρ_1,ρ_2), implies that (X,ρ_1,ρ_2) is a pairwise PN Baire space. □

Definition 4.16. A PN bitopological space (X,ρ_1,ρ_2) is a pairwise PN almost resolvable space if $\bigcup_{k=1}^{\infty}(A_k) = 1_X$, where A_k's in (X,ρ_1,ρ_2) are such that
$PNint_{\rho_1}PNint_{\rho_2}(A_k) = PNint_{\rho_2}PNint_{\rho_1}(A_k) = 0_X$.
Otherwise (X,ρ_1,ρ_2) is called a pairwise PN almost irresolvable space.

Theorem 4.17. If a PN bitopological space (X,ρ_1,ρ_2) is a pairwise PN second category space, then (X,ρ_1,ρ_2) is a pairwise PN almost irresolvable space.

Proof. Let (X,ρ_1,ρ_2) be a second category space.Then, $\bigcup_{i=1}^{\infty}(A_i) \neq 1_X$,where A_i's are PN nowhere dense sets in (X,ρ_1,ρ_2).That is, $\bigcup_{i=1}^{\infty}(A_i) \neq 1_X$,where $PNint_{\rho_1}PNcl_{\rho_2}(A_i) = 0_X = PNint_{\rho_2}PNcl_{\rho_1}(A_i)$,
Now, $PNint_{\rho_1}PNint_{\rho_2}(A_i) \subseteq PNint_{\rho_1}PNcl_{\rho_2}(A_i)$ and
$PNint_{\rho_2}PNint_{\rho_1}(A_i) \subseteq PNint_{\rho_2}PNcl_{\rho_1}(A_i)$,
implies that $PNint_{\rho_1}PNint_{\rho_2}(A_i) = 0_X$ and
$PNint_{\rho_2}PNint_{\rho_1}(A_i) = 0_X$.
Hence $\bigcup_{i=1}^{\infty}(A_i) \neq 1_X$, where $PNint_{\rho_1}PNint_{\rho_2}(A_i) = 0_X$
and $PNint_{\rho_2}PNint_{\rho_1}(A_i) = 0_X$ and therefore (X,ρ_1,ρ_2) is a pairwise PN almost irresolvable spaces. □

Proposition 4.18. Let the PN bitopological space (X,ρ_1,ρ_2) be a pairwise PN strongly irresolvable space.Then, we have the following:

(i) (X,ρ_1,ρ_2) is a pairwise PN almost irresolvable space, then (X,ρ_1,ρ_2) is a pairwise PN second category space.
(ii) (X,ρ_1,ρ_2) is a pairwise PN almost resolvable space, then (X,ρ_1,ρ_2) is a pairwise PN first category space.

Proof. (i) Let (X,ρ_1,ρ_2) be a pairwise PN almost irresolvable space.
Then, $\bigcup_{k=1}^{\infty}(A_k) \neq 1_X$, where A_k's in (X,ρ_1,ρ_2) are such that
$PNint_{\rho_1}PNint_{\rho_2}(A_k) = PNint_{\rho_2}PNint_{\rho_1}(A_k) = 0_X$.
Now, $(PNint_{\rho_1}PNint_{\rho_2}(A_k))^c = 1_X$ and
$(PNint_{\rho_2}PNint_{\rho_1}(A_k))^c = 1_X$.
Then, $PNcl_{\rho_1}PNcl_{\rho_2}((A_k)^c) = 1_X$ and $PNcl_{\rho_2}PNcl_{\rho_1}((A_k)^c) = 1_X$ and
hence $((A_k)^c)$'s are pairwise PN dense sets in (X,ρ_1,ρ_2).
Since (X,ρ_1,ρ_2) is a pairwise PN strongly irresolvable space, for the pairwise PN dense set A_k, we have
$PNcl_{\rho_1}PNint_{\rho_2}((A_k)^c) = 1_X$ and $PNcl_{\rho_2}PNint_{\rho_1}((A_k)^c) = 1_X$.

This implies that $PNint_{\rho_1}PNcl_{\rho_2}(A_k) = PNint_{\rho_2}PNcl_{\rho_1}(A_k)$
$= 0_X$ and hence A_k's are pairwise PN nowhere dense sets in (X, ρ_1, ρ_2).
Thus, $\bigcup_{k=1}^{\infty}(A_k) \neq 1_X$, where A_k's are pairwise PN nowhere dense sets in (X, ρ_1, ρ_2), implies that (X, ρ_1, ρ_2) is a pairwise PN second category space.

(ii) Let (X, ρ_1, ρ_2) be a pairwise PN almost resolvable space.
Then, $\bigcup_{k=1}^{\infty}(A_k) = 1_X$, where A_k's in (X, ρ_1, ρ_2) are such that
$PNint_{\rho_1}PNint_{\rho_2}(A_k) = PNint_{\rho_2}PNint_{\rho_1}(A_k) = 0_X$.
Now, $(PNint_{\rho_1}PNint_{\rho_2}(A_k))^c = 1_X$ and
$(PNint_{\rho_2}PNint_{\rho_1}(A_k))^c = 1_X$.
Then, $PNcl_{\rho_1}PNcl_{\rho_2}((A_k)^c) = 1_X$ and $PNcl_{\rho_2}PNcl_{\rho_1}((A_k)^c) = 1_X$ and hence $((A_k)^c)$'s are pairwise PN dense sets in (X, ρ_1, ρ_2).Since (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable space, for the pairwise PN dense set A_k, we have
$PNcl_{\rho_1}PNint_{\rho_2}((A_k)^c) = 1_X$ and $PNcl_{\rho_2}PNint_{\rho_1}(A_k)^c = 1_X$.
This implies that $PNint_{\rho_1}PNcl_{\rho_2}(A_k) = PNint_{\rho_2}PNcl_{\rho_1}(A_k)$
$= 0_X$ and hence A_k's are pairwise PN nowhere dense sets in (X, ρ_1, ρ_2).
Thus, $\bigcup_{k=1}^{\infty}(A_k) = 1_X$, where A_k's are pairwise PN nowhere dense sets in (X, ρ_1, ρ_2), implies that (X, ρ_1, ρ_2) is a pairwise PN first category space. □

Theorem 4.19. If $\bigcup_{k=1}^{\infty}(A_k) = 1_X$, where s A_k's are pairwise PN σ-nowhere dense sets in a PN bitopological space (X, ρ_1, ρ_2), then (X, ρ_1, ρ_2) is a pairwise PN almost resolvable space.

Definition 4.20. A PN bitopological space (X, ρ_1, ρ_2) is called pairwise PN σ-first category space if the PN 1_X is a pairwise PN σ-first category set in (X, ρ_1, ρ_2).That is, $1_X = \bigcup_{i=1}^{\infty}(A_i)$, where A_i's are pairwise PN σ-nowhere dense sets in (X, ρ_1, ρ_2).Otherwise, (X, ρ_1, ρ_2) will be called a pairwise PN σ-second category space.

Proposition 4.21. If (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable and pairwise PN σ-first category space, then (X, ρ_1, ρ_2) is a pairwise PN first category space.

Proof. Let (X, ρ_1, ρ_2) be a pairwise PN σ-first category space.
Then, $\bigcup_{k=1}^{\infty}(A_k) = 1_X$, where A_k's are pairwise PN σ-nowhere dense sets in (X, ρ_1, ρ_2).
Since (X, ρ_1, ρ_2) is a pairwise PN strongly irresolvable space, by theorem 4.7, then A_k's are pairwise PN nowhere dense sets in (X, ρ_1, ρ_2).
Hence $\bigcup_{k=1}^{\infty}(A_k) = 1_X$, where the A_k's arepairwise PN nowhere dense sets in (X, ρ_1, ρ_2), implies that (X, ρ_1, ρ_2) is a pairwise PN first category space. □

Proposition 4.22. If (X, ρ_1, ρ_2) is a pairwise PN Baire and pairwise PN strongly irresolvable space, then $PNcl_{\rho_i}(\bigcap_{k=1}^{\infty}(A_k)) = 1_X (i = 1, 2)$ where B_k's are pairwise PN dense in (X, ρ_1, ρ_2).

Proof. Let (X, ρ_1, ρ_2) be a pairwise PN Baire space.
Then, $PNint_{\rho_i}(\bigcup_{k=1}^{\infty}(A_k)) = 0_X, (i = 1, 2)$, where A_k's are pairwise PN nowhere dense sets in (X, ρ_1, ρ_2).
Since A_k's are pairwise PN nowhere dense sets in (X, ρ_1, ρ_2),by theorem 4.2, $(A_k)^c$'s are pairwise PN dense sets in (X, ρ_1, ρ_2).
Now $PNint_{\rho_i}(\bigcup_{k=1}^{\infty}(A_k)) = 0_X$, implies that $(PNint_{\rho_i}(\bigcup_{k=1}^{\infty}(A_k)))^c = 1_X$.
Then, we have $PNcl_{\rho_i}(\bigcap_{k=1}^{\infty}(A_K)^c) = 1_X$.Let $(A_k)^c = B_k$.
Then $PNcl_{\rho_i}(\bigcap_{k=1}^{\infty}(B_k)) = 1_X$, where B_k's are pairwise PN dense sets in (X, ρ_1, ρ_2). □

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Decision Making on Teachers' adaptation to Cybergogy in Saturated Interval- valued Refined Neutrosophic overset /underset /offset Environment

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Abstract

Neutrosophic overset, neutrosophic underset and neutrosophic offset introduced by Smarandache are the special kinds of neutrosophic sets with values beyond the range [0,1] and these sets are pragmatic in nature as it represents the real life situations. This paper introduces the concept of saturated refined neutrosophic sets and extends the same to the special kinds of neutrosophic sets. The proposed concept is applied in decision making on Teacher's adaptation to cybergogy. The decision making environment is characterized by different types of teachers, online teaching skills and various training methods. Fuzzy relation is used to match the most suitable method to the different kinds of teachers with the intervention of saturated interval valued neutrosophic refined oversets, offsets and undersets. The results obtained by applying the notion of saturated refined sets using various distance measures represent the effect of training methods on teacher's adaptation to learner-centred teaching methods, which certainly give space to gain many insights on the relationship between quality of training and teacher's adaptation rate. The proposed concept has wide scope and few limitations.

Keywords: neutrosophic oversets, neutrosophic offsets, neutrosophic offsets, refined sets, saturated, interval-valued sets, cybergogy

1.Introduction

Decision making is an indispensable activity but the environment is highly characterized with uncertainty. The concept of fuzzy set introduced by Lofti.A.Zadeh [1] plays a vital role in tackling such uncertain and imprecise situations.

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Fuzzy sets differ from crisp sets by membership functions and membership values. The elements of crisp sets contain binary membership values i.e either 1 or 0, it doesn't deal with intermediate values. Fuzzy sets overcome this short coming with the inclusion of intermediate values and extending the range of values from {0,1} to [0,1]. Fuzzy sets are highly comprehensive and inclusive in nature. Fuzzy sets are extensively used to handle complex systems and control as these sets possess high rate of industrial applications. Fuzzy sets are extended to intuitionistic sets by Atanssov[2] with the introduction of non-membership function to membership function. The elements of intuitionistic sets possess both membership and non-membership values ranging from [0,1]. The hesitancy margin is calculated by subtracting the sum of membership and non-membership values from 1. In intuitionistic sets the hesitancy margin is dependent on membership and non-membership values. Intuitionistic sets and various forms of it are widely used in multiple attribute decision making. Khan et al [3] used Interval-valued Pythagorean fuzzy GRA method for multiple-attribute decision making with incomplete weight information. Zhuosheng Jia et al[4] used interval valued intuitionistic fuzzy sets in multiple attribute group decision making method TOPSIS. Intuitionistic sets are further extended to neutrosophic sets by Smarandache[5] and these sets have truth membership functions, indeterminacy functions and non-membership functions. The elements of neutrosophic sets are triplets with independent truth, indeterminacy and false membership values ranging from [0,1]. Neutrosophic sets are widely used in multiple attribute decision making. Abdel-Baset et al [6,7] developed multi criteria decision making method with neutrosophic representation in evaluating green supply chain management practices and in sustainable supplier selection. Hu et al [8] also contributed to neutrosophic decision making on the selection of doctors.Nada A. Nabeeh et al [9] proposed a hybrid approach of neutrosophic with MULTIMOORA in application of personnel selection. Ajay et al [10] developed the single -valued triangular neutrosophic approach of decision making on multi objectives based on ratio analysis.Sahidul Islam et al [11] formulated neutrosophic goal programming approach to a green supplier selection model with quantity discount. Mullai.M et al [12] used neutrosophic intelligent energy efficient routing for wireless ad-hoc network based on multi-criteria decision making. Abdel Nasser et al [13] proposed an integrated neutrosophic and TOPSIS for evaluating airline service quality. Neutrosophic hypersoft sets are also used in decision making. Muhammad Saqlain et al [14] presented the applications of neutrosophic hypersoft sets in TOPSIS using accuracy function. Surapati et al[15] developed Multi-level linear programming problem with neutrosophic numbers. Ajay et al [16,17] discussed decision making techniques based on bipolar neutrosophic sets, neutrosophic cubic fuzy sets , Chakravarthy et al [18,19] expounded the implications of cyclindrical and pentagonal neutrosophic numbers in networking and mobile communication respectively . Deli et al [20,21] proposed multi attribute decision making models based on weighted geometric operators and two centroid point for single valued triangular neutrosophic number. Neutrosophic graphs are also widely used in decision making. Juanjuan et al [22] developed a multi attribute decision making model using single valued neutrosophic graphs. Dragisa et al [23] proposed a novel approach of assessing the reliability of the data in decision making. Shahzaib et al [24] framed a decision making model to select agroculture land using neutrosophic information. Muhammad et al [25] developed auto car decision making model using Bipolar Neutrosophic Soft Sets. Philippe [26] has also discussed the neutrosophical representations in cognitive dimension. The neutrosophic sets are extensively applied in multi criteria decision making.

Smarandache [27] introduced neutrosophic oversets, offsets and undersets which are the special kinds of neutrosophic sets with values beyond [0,1]. Overset is characterized with membership values greater than 1, underset is characterized with membership values less than 0 and the combination of both these sets is offset. Smarandache justified the practical implications of these special kinds of sets with real life illustrations. These kinds of neutrosophic sets highly influenced and motivated us to propose a fuzzy relational decision making model with saturated refined interval- valued neutrosophic oversets, undersets and offsets based on application of refined neutrosophic sets in medical diagnosis by Deli et al [28]. Smarandache conceptualized n-valued refined neutrosophic sets and these sets are used in decision making model of medical diagnosis. Broumi [29] extended the model of Deli et al by applying correlation measure. Various distance measures are used to make optimal decisions without changing the neutrosophic representations. In their model relation between symptoms and diseases was represented by neutrosophic sets; relation between patients and symptoms was represented by refined neutrosophic sets over certain interval period of time. In this decision making model the representation of the symptoms of the patients varies from time to time. But on

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profound analysis, the effects of treatment on the status and the degree of symptoms lack representation. This deficit in the decision maing model paved the way for developing a novel decision making model with new kind of representations.The same model is extended to fuzzy relation decision making model on teacher's adaptation to cybergogy in this research work. A relation between digital teaching skills and training methods is represented by neutrosophic sets and the relation between different kinds of teachers and the acquisition of digital skills after continuous stages of training is represented by refined neutrosophic oversets, undersets and offsets. Such kinds of representations are made to reflect the impact of training on skill acquisition rate by the teachers. The degree of digital skill acquisition by the teacher greatly depends on the personal interest, trainer's approach and training environment. The self- interest of the teachers may induce them to spend additional time other than the specified training time; also the disinterest of the teachers or dislike of trainer's approach may make them to refrain from the training and their participation rate is disturbed. At such circumstances refined neutrosophic oversets, underset and offset are used to represent such impacts. Also a new concept of saturated refined sets is introduced in this paper. The refined neutrosophic overset, underset and offset values remain to settle to a particular value over a consecutive period of time then it is called as saturated. The existences of situations where the degree of digital skill acquisition is confined and attained the maximum value and also there is no chance of further change over a period of training can be represented by saturated refined neutrosophic sets. The apt method of training to different kinds of teachers is determined by using hamming distance, normalized hamming distance, Euclidean distance and normalized Euclidean distance measures. The practical implications of neutrosophic overset, underset and offset are not explored to the best of the knowledge and so this research work will certainly fill the gap and it is intended to do so.

The paper is organized as follows: section 2 presents the basic definitions; section 3 describes about saturated refined neutrosophic sets; section 4 consists of the application of the proposed model; section 5 discusses the results and the last section concludes the work.

2. Preliminaries

Definition 2.1 [27]

Let X be an universe of discourse, A neutrosophic set A in X is expressed by A=$\{< X; T_A(x), I_A(x), F_A(x) >/x \in X\}$and T,I,F:X→]$^-$0:1$^+$[where T,I,F are the degree of membership(True), the indeterminacy and degree of non-membership(False) respectively, and 0≤ $T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.2 [27]

Let X be the universe of discourse with a generic element in X is denoted by x. An interval valued neutrosophic set (IVNS) A in X is defined by A={ x,<$[T_A^L(x), T_A^U(x)]$, $[I_A^L(x), I_A^U(x)]$, $[F_A^L(x), F_A^U(x)] >;\ x \in X\}$

where T_A, I_A, F_A are the truth membership function, indeterminacy membership function, falsity membership function respectively.For each point x in X, We have $[T_A^L(x), T_A^U(x)]$, $[I_A^L(x), I_A^U(x)]$,$[F_A^L(x), F_A^U(x)]$⊆[0,1] with the condition 0≤ $T_A^U(x) + I_A^U(x) + F_A^U(x) \leq 3$.

Definition2.3 [27]

Let U be a universe of discourse. A neutrosophic refined set (NRS) A on U can be defined as follows

A={<x,<$T_A^1(x), T_A^2(x) \dots T_A^P(x)), \big(I_A^1(x), I_A^2(x) \dots I_A^P(x)\big), (F_A^1(x) F_A^2(x) \dots F_A^P(x) > x \in U\}$ and

0≤ $T_A^i(x) + I_A^i(x) + F_A^i(x) \leq 3, (i = 1,2 \dots P)andT_A^1(x) \leq T_A^2(x) \leq \cdots \leq T_A^P(x)\ \ for\ any$

$x \in A$,$T_A^i(x),\ I_A^i(x),\ F_A^i(x), i = 1,2 \dots P$is the truth membership sequence,indeterminancy membership sequence and falsity membership sequence of the element x respectively.

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Definition 2.4 [27]

Let U be the universe of discourse. A neutrosophic set A_1 in U which consist the membership function T(x),I(x),F(x) that define true, Indeterminacy and falsity respectively, of a generic element x$\in U$,

$T_A(x), I_A(x), F_A(x): U \to [0, \varphi] where\ 0 < 1 < \varphi, and\ \varphi$ is called over limit.

A single valued neutrosophic over set A_1 is defined as

A_1={x,<$T_A(x); I_A(x); F_A(x)$>x$\in U$} such that in the neutrosophic components contains there exist atleast one element in A_1 is >1 and no element is < 0.

Definition 2.5[27]

Let U be the universe of discourse. A neutrosophic set A_2 in U which consist the membership function T(x), I(x), F(x) that define true, Indeterminacy and falsity respectively, of a generic element x$\in U$,

$T_A(x), I_A(x), F_A(x): U \to [\psi, 1] where\ \psi < 0 < 1, and\ \psi$ is called under limit.

A single valued neutrosophic underset A_2 is defined as

A_2={x,<$T_A(x); I_A(x); F_A(x)$>x$\in U$}

such that in the neutrosophic components contains there exist atleast one element in A_2 is <0 and no element is >1

Definition 2.6 [27]

Let U be the universe of discourse. A neutrosophic set A_3 in U which consist the membership function T(x),I(x),F(x) that define true, Indeterminacy and falsity respectively, of a generic element x$\in U$,

$T_A(x), I_A(x), F_A(x): U \to [\psi, \varphi] where\ \psi < 0 < 1 < \varphi, and\ \psi$ is called under limit while φ is called over limit, $T_A(x), I_A(x), F_A(x) \in [\psi, \varphi]$.The neutrosophic single-valued offset A_3 is defined by

A_3 = {x, <$T_A(x); I_A(x); F_A(x)$> x$\in U$} such that in the neutrosophic components contains there is atleast one element is >1 and atleast another is < 0.

Definition 2.7 [27]

Let U be the universe of discourse. A neutrosophic set A_1 in U which consist the membership function T(x), I(x), F(x) that define true, Indeterminacy and falsity respectively, of a generic element x$\in U$,

$T_A(x), I_A(x), F_A(x): U \to P([0, \varphi]) where\ 0 < 1 < \varphi, and\ \varphi$ is called over limit,

$T_A(x), I_A(x), F_A(x) \subseteq [0, \varphi], and\ P([0, \varphi])$ is the set of all subsets of $[0, \varphi]$.An interval valued neutrosophic overset A_1 is defined as A_1={x,<$T_A(x); I_A(x); F_A(x)$>x$\in U$} such that in the neutrosophic component contains there is atleast one is partially or totally above 1 and no element has partially or totally below 0.

Definition 2.8 [27]

Let U be the universe of discourse. A neutrosophic set A_2 in U which consist the membership function T(x), I(x), F(x) that define true, Indeterminacy and falsity respectively, of a generic element x$\in U$,

$T_A(x), I_A(x), F_A(x): U \to P([\psi, 1]) where\ \psi < 0 < 1, and\ \psi$ is called under limit.

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$T_A(x), I_A(x), F_A(x) \subseteq [\psi, 1], and\ P([\psi, 1])$ is the set of all subsets of $[\psi, 1]$.An interval valued neutrosophic overset A_2 is defined as A_2={x,<$T_A(x); I_A(x); F_A(x)$>x∈ $U\}$ such that in the neutrosophic component contains there is atleast one is partially or totally below 0 and no element has partially or totally above 1.

Definition 2.9 [27]

Let U be the universe of discourse. A neutrosophic set A_3 in U which consist the membership function T(x),I(x),F(x) that define true, indeterminacy and falsity respectively, of a generic element x∈ U,

$T_A(x), I_A(x), F_A(x): U \rightarrow P[\psi, \varphi] where\ \psi < 0 < 1 < \varphi, and\ \psi$ is called under limit while φ is called over limit, $T_A(x), I_A(x), F_A(x) \subseteq P\ [\psi, \varphi]$ and $P\ [\psi, \varphi]\ is$ the set of all subsets of $[\psi, \varphi]$

An interval valued neutrosophic offset A_3 is defined as A_3 = {x, <$T_A(x); I_A(x); F_A(x)$> x∈ $U\}$ such that in the neutrosophic components contains atleast one is partially or totally above 1 and atleast another is partially or totally below 0.

Definition: 2.10 [17]

Let A,B ∈ $IVNRS(U)$.Then

1.Hamming distance between A and B is denoted as d_H(A,B) and is defined by

$$d_H(A,B) = \frac{1}{6}\sum_{j=1}^{P}\sum_{i=1}^{n}(|T_A^L(x_i) - T_B^L(x_i)| + |T_A^U(x_i) - T_B^U(x_i)| + |I_A^L(x_i) - I_B^L(x_i)| + |I_A^U(x_i) - I_B^U(x_i)| + |F_A^L(x_i) - F_B^L(x_i)| + |F_A^U(x_i) - F_B^U(x_i)|$$

2.Normalized hamming distance between A and B is denoted as d_{NH}(A,B) and is defined by

$$d_H(A,B) = \frac{1}{6nP}\sum_{j=1}^{P}\sum_{i=1}^{n}(|T_A^L(x_i) - T_B^L(x_i)| + |T_A^U(x_i) - T_B^U(x_i)| + |I_A^L(x_i) - I_B^L(x_i)| + |I_A^U(x_i) - I_B^U(x_i)| + |F_A^L(x_i) - F_B^L(x_i)| + |F_A^U(x_i) - F_B^U(x_i)|$$

3.Euclidean distance between A and B is denoted as d_E(A,B) and is defined by

$$d_E(A,B) = \frac{1}{6}\sum_{j=1}^{P}\sum_{i=1}^{n}\{(|T_A^L(x_i) - T_B^L(x_i)|^2 + |T_A^U(x_i) - T_B^U(x_i)|^2 + |I_A^L(x_i) - I_B^L(x_i)|^2 + |I_A^U(x_i) - I_B^U(x_i)|^2 + |F_A^L(x_i) - F_B^L(x_i)|^2 + |F_A^U(x_i) - F_B^U(x_i)|^2)\}^{\frac{1}{2}}$$

4.Normalized Euclidean distance between A and B is denoted as d_{NE}(A,B) and is defined $d_E(A,B) = \frac{1}{6nP}\sum_{j=1}^{P}\sum_{i=1}^{n}\{(|T_A^L(x_i) - T_B^L(x_i)|^2 + |T_A^U(x_i) - T_B^U(x_i)|^2 + |I_A^L(x_i) - I_B^L(x_i)|^2 + |I_A^U(x_i) - I_B^U(x_i)|^2 + |F_A^L(x_i) - F_B^L(x_i)|^2 + |F_A^U(x_i) - F_B^U(x_i)|^2)\}^{\frac{1}{2}}$

3. Saturated Refined Neutrosophic sets

Irfan Deli et al [28] presented the properties and various operations of neutrosophic refined sets. An element of neutrosophic refined set has a sequence of truth, indeterminacy and falsity membership values. In the model

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proposed by Irfan Deli et al [28] the symptoms of the patients at three different intervals of time are presented as neutrosophic refined sets. But in reality if the patients are undergoing treatment and the symptoms are checked at different intervals of time, suppose if a patient gets cured and gets back to normal conditions, then the symptoms of the disease are nil and it takes same values if testing of symptoms takes place at consecutive period of time. At this junction the membership values gets saturated, and this instance is the origin of saturated refined neutrosophic sets.

Let U be a universe of discourse. A neutrosophic saturated refined set (NSRS) A on U can be defined as follows

A={<x, $(T_A^1(x), T_A^2(x) \ldots, T_A^P(x), T_A^P(x)), \left(I_A^1(x), I_A^2(x) \ldots I_A^P(x), I_A^P(x)\right), (F_A^1(x) F_A^2(x) \ldots F_A^P(x), F_A^P(x) > x \in$ $U$and $0\leq T_A^i(x) + I_A^i(x) + F_A^i(x) \leq 3, (i = 1,2 \ldots P)andT_A^1(x) \leq T_A^2(x) \leq \cdots \leq T_A^P(x)$ $for\ any$

$x \in A, T_A^i(x),\ I_A^i(x),\ F_A^i(x), i = 1,2 \ldots P$is the truth membership sequence, indeterminacy membership sequence and falsity membership sequence of the element x respectively.

Let U be a universe of discourse. A interval – valued saturated refined neutrosophic set A on U can be defined as follows

A={<x, $\left(\left[T_A^{L_1}(x), T_A^{U_1}(x)\right], \left[T_A^{L_2}(x), T_A^{U_2}(x)\right], \ldots . T_A^p(x)\right), \left(\left[I_A^{L_1}(x), I_A^{U_1}(x)\right], \left[I_A^{L_2}(x), I_A^{U_2}(x)\right], \ldots . I_A^p(x)\right),$ $(\left[F_A^{L_1}(x), F_A^{U_1}(x)\right], \left[F_A^{L_2}(x), F_A^{U_2}(x)\right], \ldots . F_A^p(x)) > x \in U$and $0\leq T_A^i(x) + I_A^i(x) + F_A^i(x) \leq 3, (i = 1,2 \ldots P)andT_A^1(x) \leq T_A^2(x) \leq \cdots \leq T_A^P(x)$ $for\ any$

$x \in A, T_A^i(x),\ I_A^i(x),\ F_A^i(x), i = 1,2 \ldots P$is the truth membership sequence, indeterminacy membership sequence and falsity membership sequence of the element x respectively.

Remark:

1. If any of the membership values is saturated it is partial in nature and it is also a saturated refined set.
2. The saturated refined neutrosophic sets can be extended to overset, underset and offset.
3. The interval – valued refined neutrosophic sets are also extended to saturated interval- valued refined neutrosophic sets and the saturated values varies from interval sets to single valued sets over a period of time.

4. Application of the proposed decision making model

A decision making model together with fuzzy relational matrix and saturated refined neutrosophic overset, underset and offset is validated with the following illustration.

Decision Making Environment

Presently COVID – 19 has brought a paradigm shift in teaching and learning process, the teaching fraternity is expected to possess digital teaching skills to face the post quarantine period. The developing nations have begun to encourage online educational system with the motive of unlocking learning during lock down. In this juncture the teachers are categorized based on their attributes and exposed to different kinds of training method to foster the acquisition of digital skills. The ultimate aim of this decision making model is to determine the suitable training method to the different kinds of teachers. This training programme is conducted to train the teachers to acquire online teaching skills. The expected outcome is enhancement of teacher's online teaching skills. The effectiveness of the programme is evaluated based on certain attributes and these attributes duly play crucial role in the enhancement of teacher's online teaching skills.

The attributes are

A1 Trainer’s efficiency- Refers to mastery

A2 Teacher’s interest

A3 Teacher’s duration of participation – present throughout the sessions

A4 Teacher’s grasping ability – how quick they understand

A5 Trainer’s Approach – Refers to inter personal relationship/ social skills

The teachers are made to undergo four phases of training namely I, II, III, IV and they are grouped into four categories and their characteristic features are presented in Table 4.1

Table 4.1 Types of Teachers & Attributes

Types of Teachers	**Characterization**
T1	Encouraging,Motivating,Systematic,Holistic
T2	Optimistic,creative,interactive,Facllitative
T3	Pragmatic,realistic,joyful,flexible
T4	Weak commitment, Projective,Low professional ,Resistant to change

The training to teachers are given using the following modes such as Self- paced learning, Blended learning, Adaptive learning, Virtual classes. The digital skills that are focussed in this training programme are Online skills, Digital literacy skills, Administrative skills of Learning Management System (LMS), Technology skills, Organization skills. The relation between digital skills and training methods are presented in Table 4.2

Table 4.2 Relation between skills and methods

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	Blended Mode	Self- paced Mode	Adaptive Mode	Virtual Mode
Online Skills	([.45,.6],[0.2,.3],[0.39,.46])	([0.66,.8],[0.43,.5 7],[0.2,.38])	([0.63,.75],[0.49,. 7],[0.3,.4])	([0.43,.58],[0.33,. 5],[0.61,.68])
Digital literacy Skills	([0.5,.61],[0.1,.39],[.35,0.48])	([0.43,.6],[0.15,.2 5],[0.38,.5])	([0.13,.31],[0.5,.7 1],[0.44,.47])	([0.3,.39],[0.1,.43],[0.51,.62])
Administrative skills of LMS	([0.5,.65],[0.21,.4 5],[0.36,.57])	([0.45,.68],[0.2,.3 7],[0.64,.71])	([0.46,.62],[0.2,.3 9],[0.24,.29])	([0.65,.78],[0.32,. 39],[0.53,.62])
Technology Skills	([0.4,.63],[0.1,.25], [0.69,.71])	([0.44,.5],[0.61,.6 9], [0.25,.45])	([0.27,.3],[0.53,.6], [0.39,.48])	(0.51,.59],[0.35,. 4], [0.23,.33])
Organization skills	([0.45,.5],[0.61,.6 9], [0.15,.2])	([0.3,.65],[0.42,.5 2], [0.18,.28])	([0.33,.45],[0.45,. 5],[0.25,.3])	([0.4,.53],[0.2,.35],[0.13,.25])

The degree of acquisition rate of digital teaching skills is presented as saturated refined interval-valued neutrosophic overset, underset and offset in Table 4.3.

Table 4.3 Relation between Teachers and Skill acquisition

	Online Skills	**Digital literacy Skills**	**Administrative skills of LMS**	**Technology Skills**	**Organization skills**
T1	([0.7,0.8],[0.3,0.4], [0.5,0.7]) ([0.75,0.85],[0.41,0. 5],[0.45,0.6]) ([0.79,0.95],[0.45,0. 58],0.4)	[(0.6,0.7],[0.2,0.3], [0.5,0.6]) ([0.65,0.75],[0.28,0 .35],[0.43,0.48]) ([0.78,0.88],[0.36,0 .4],0.31) ([0.89,1.1],[0.37,0. 41],0.31)	([0.5,0.6],[0.2,0.3],[0.1,0.3]) ([0.55,0.67],[0.31,0. 43],[0.1,0.28]) (0.61,[0.33,0.44],[.0 .1,0.15]) (0.61,[0.35,0.44],[- .0.1,0.12])	([0.3,0.4],[0.7,0.8],[0.4,0.6]) ([0.37,0.45],[0.81,0. 93],[0.52,0.58]) ([0.39,0.48],1.3,[0.5 3,0.56]) ([0.43,0.52],1.3,[0.5 4,0.55])	([0.4,0.5],[0.2,0. 3],[0.03,0.05]) ([0.46,0.57],[0.2 6,0.35],[0.01,0.0 3]) (0.52,[0.28,0.37], [0.01,0.02])

	([0.9,1.2],[0.48,0.59],0.4)				([0.52,[0.31,0.38],[-0.01,0.01])
T2	([0.6,0.7],[0.3,0.4],[0.7.0.8]) ([0.75,0.83],[0.33.0.46],[0.71,0.75]) ([0.83,0.95],[0.4,0.48],0.68) ([0.96,1.3],[0.44,0.54],0.68)	([0.5,0.6],[0.6,0.7],[0.1,0.2]) ([0.56,0.61],[0.68,0.73],[0.1,0.15]) ([0.59,[0.71,0.75],[0.1,0.12]) ([0.59,[0.72,0.77],[-0.1,0.11])	([0.4,0.5],[0.8,0.9],[0.3,0.4]) ([0.47.0.58],[0.88,0.97],[0.3,0.35]) ([0.55,0.63],1.13,[0.3,0.34]) ([0.55,0.78],1.13,[0.3,0.31])	([0.7,0.8],[0.3,0.4],[0.2,0.3]) ([0.75,0.83][0.35,0.43],[0.1,0.2]) ([0.86,0.95],0.39,[0.05,0.1) ([0.96,1.1],0.39,[-0.01,0.04])	([0.3,0.4],[0.7,0.8],[0.5,0.6]) ([0.36,0.46],[0.79,0.89],[0.5,0.55]) ([0.37,0.53],[0.89,0.99],0.52) ([0.45,0.56],[0.95,1.1],0.52)
T3	([0.8,0.9],[0.2,0.3],[0.4,0.5]) ([0.86,0.98],[0.3,0.41],[0.4,0.45]) (1.1,[0.38,0.49],[0.35,0.4) (1.1,[0.42.0.51],[0.35,0.38])	([0.4,0.5],[0.8,0.9],[0.5,0.6]) ([0.47,0.57],[0.83,0.93],[0.45,0.57]) ([0.55,0.61],1.2,[0.45,0.51]) ([0.55,0.62],1.2,[0.45,0.49])	([0.3,0.4],[0.2,0.3],[0.1,0.2]) ([0.38,0.47],[0.26,0.37],[0.1,0.15]) (0.45,[0.28,0.39],[0.1,0.12]) (0.45,[0.28,0.42],[-0.1,0.1])	([0.5,0.6],[0.7,0.8],[0.3,0.4]) ([0.57,0.68],[0.85,0.91],[0.2,0.3]) (0.63,[0.91,0.99],[0.2,0.25]) (0.63,[0.98,1.2],[0.21,0.23])	([0.3,0.4],[0.4,0.5],[0.2,0.3]) ([0.35,0.46],[0.47,0.57],[0.1,0.2]) (0.43,0.52,[0.1,0.15]) (0.43,0.52,[-0.1,0.12])
T4	([0.3,0.4],[0.5,0.6],[0.2,0.3]) ([0.41,0.49],[0.6.0.67],[0.1,0.2]) ([0.45,0.53],0.61,[0.1,0.15]) ([0.45,0.55],0.61,	([0.5,0.6],[0.8,0.9],[0.6,0.7]) ([0.52,0.61],[0.88,0.98],[0.5,0.6]) ([0.63,0.64],1.3,[0.5,0.55])	([0.7,0.8],[0.5,0.6],[0.36,0.57]) ([0.8,0.91],[0.55,0.65],[0.36,0.55]) ([0.88,0.99],0.61,[0.2,0.25])	([0.2,0.3],[0.1,0.27],[0.1,0.2]) ([0.4,0.63],[0.1,0.25],[0.1,0.13]) ([0.46,0.63],0.26,[0.05,0.1])	([0.3,0.4],[0.8,0.9],[0.5,0.6]) ([0.45,0.53],[0.88,0.91],[0.45,0.5]) (0.45,[0.98,1.1],0.41)

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	[-0.1,0.11])	([0.65,0.71],1.3,[0.48,0.52])	([0.98,1.1],0.61,[0.2,0.23])	([0.47,0.63],0.26,[-0.05,0.05])	(0.45,[0.99,1.2],0.41)

The normalized Hamming distance is used to determine the most suitable training method to teachers and the values are presented in Table 4.4.

Table 4.4 Normalized Hamming Distance between Teachers and Methods

	Blended Mode	**Self- paced Mode**	**Adaptive Mode**	**Virtual Mode**
T1	0.214	**0.183**	0.194	0.217
T2	0.286	0.299	0.28	**0.279**
T3	0.245	0.199	**0.185**	0.246
T4	**0.254**	0.286	0.26	0.27

The results obtained by Hamming distance, Euclidean and Normalized Euclidean distance methods are presented in Table 4.5,4.6 and 4.7 respectively

Table 4.5 Hamming Distance between Teachers and Methods

	Blended Mode	Self-paced Mode	Adaptive Mode	Virtual Mode
T1	0.858	**0.73**	0.774	0.866
T2	1.142	1.174	1.122	**1.117**
T3	0.999	0.798	**0.741**	0.986
T4	**1.014**	1.143	1.037	1.076

Table 4.6 Euclidean Distance between Teachers and Methods

	Blended Mode	Self-paced Mode	Adaptive Mode	Virtual Mode
T1	0.115	**0.095**	0.096	0.12
T2	0.129	0.13	0.128	**0.127**
T3	0.131	0.112	**0.0876**	0.124

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T4	**0.129**	0.132	0.131	0.133

Table 4.7 Normalized Euclidean Distance between Teachers and Methods

	Blended Mode	Self-paced Mode	Adaptive Mode	Virtual Mode
T1	0.0289	**0.0239**	0.024	0.0274
T2	0.0321	0.0325	0.0323	**0.0317**
T3	0.0328	0.028	**0.0219**	0.031
T4	**0.0277**	0.033	0.0324	0.0332

Discussion

Table 4.4,4.5,4.6 &4.7 clearly presents the most suitable training method to various kinds of teachers. The lowest distance gives the apt method. Self- paced mode is suitable to type I teachers; Virtual mode to type II teachers; Adaptative mode to type III teachers and blended mode to type IV teachers. This optimal relation between teachers and methods are highly pragmatic as it has incorporated the influence of external and internal factors of the training programme. The various methods of distance measures are used to determine the feasible method of teaching and on comparative analysis, the results obtained by using the different methods, are same. The proposed decision making model with saturated refined neutrosophic sets of different kinds can be extended further with other representations of neutrosophic sets, also other kinds of distance measures can be applied to find the optimal method of teaching. This model also has certain limitations as neutrosophic oversets, undersets and offsets of representations are used only specifically and these special kinds of representations cannot be applied at all circumstances. This decision -making model caters to particular needs.

Conclusion

In this research work the concept of saturated refined neutrosophic sets, interval –valued saturated refined neutrosophic sets and its extension to neutrosophic overset, underset and offset are proposed. A decision making model with fuzzy relational matrix and saturated refined neutrosophic overset, underset and offset is proposed in this paper. The model is validated with a real life application. This research work will certainly enlighten the researchers to explore in deep about the concepts of neutrosophic overset, underset and offset. The profound extension of these concepts will disclose new portals of neutrosophic research.

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Introduction to AntiGroups

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Abstract

The notion of AntiGroups is formally presented in this paper. A particular class of AntiGroups of type-AG[4] is studied with several examples and basic properties presented. In AntiGroups of type-AG[4], the existence of an inverse is taking to be totally false for all the elements while the closure law, the existence of identity element, the axioms of associativity and commutativity are taking to be either partially true, partially indeterminate or partially false for some elements. It is shown that some algebraic properties of the classical groups do not hold in the class of AntiGroups of type-AG[4]. Specifically, it is shown that intersection of two AntiSubgroups is not necessarily an AntiSubgroup and the union of two AntiSubgroups may be an AntiSubgroup. Also, it is shown that distinct left(right) cosets of AntiSubgroups of AntiGroups of type-AG[4] do not partition the AntiGroups; and that Lagranges' theorem and fundamental theorem of homomorphisms of the classical groups do not hold in the class of AntiGroups of type-AG[4].
Keywords: NeutroGroup, AntiGroup, AntiSubgroup, AntiQuotientGroup, AntiGroupHomomorphism.

1 Introduction

Neutrosophic logic (NL) introduced by Smarandache in 1995 is an alternative to the existing classical logics and the generalization of fuzzy logic (FL) of Zadeh [12] and intuitionistic fuzzy logic (IFL) of Atanassov [5]. Neutrosophic logic is a non-classical logic that can be used as a mathematical tool to model situations characterized by uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction etc. In neutrosophic logic [11], each proposition is estimated to have percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F where (T, I, F) are standard or non-standard subsets of the non-standard interval $]^{-}0, 1^{+}[$, where $n_{\inf} = \inf T + \inf I + \inf F \geq^{-} 0$, and $n_{\sup} = \sup T + \sup I + \sup F \leq 3^{+}$. Statically, (T, I, F) are subsets but dynamically, they a functions/operators depending on many known or unknown parameters. In NL, if $<$ A $>$ is an idea, or proposition, theory, law, axiom, event, concept, entity etc., there correspond $<$ Non-A $>=<$ Anti-A $>$ which is the opposite of $<$ A $>$ and $<$ Neut-A $>$ which stands for what is neither $<$ A $>$ nor $<$ Anti-A $>$, that is neutrality in between the two extremes. Consequently in NL, it is possible to have the triad ($<$ A $>$, $<$ Neut-A $>$, $<$ Anti-A $>$). The non-restriction in NL allows for paraconsistent, dialetheist, and incomplete information to be characterized. This special and unique feature of NL has made it applicable in solving problems involving uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction etc. arising from science, social science, engineering, technology, computer science, artificial intelligence, ICT, robotics etc.

The indeterminacy factor I in NL is fundamental in the formulation and establishment of any neutrosophic algebraic structure. Given any classical algebraic structure $(X, *)$, a structure $X(I) =< X \cup I >$ generated by X and I under the binary operation $*$ of X is called a neutrosophic algebraic structure with its name derived from the name of X. For instance, if X is a group, then $X(I)$ is called a neutrosophic group. Since I^{-1}, the inverse of I does not exist, finding x^{-1}, the inverse of any neutrosophic element $x \in X(I)$ becomes difficult and impossible. Consequently, algebraic manipulations of the elements of $X(I)$ become restrictive. The recent introduction of the concepts of NeutroStructures and AntiStructures have lessened the restrictiveness of the algebraic manipulations of the elements of neutrosophic algebraic structures imposed by the neutrosophic element I.

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In any classical algebraic structure $(X, *)$, the of composition of the elements with respect to the binary operation $*$ is well defined for all the elements of X that is, $x * y \in X \;\; \forall x, y \in X$. All the axioms like associativity, commutativity, distributivity, monotonicity etc. defined on X with respect to $*$ are totally true for all the elements of X. The compositions of elements of X this way are restrictive and do not reflect the reality. They do not give rooms for compositions that are either partially defined, partially undefined (indeterminate), and partially outerdefined or totally outerdefined with respect to $*$. However in the domain of knowledge, science and reality, the law of composition and axioms defined on X may either be only partially defined (partially true), or partially undefined (partially false), or totally undefined (totally false) with respect to the binary operation $*$. In 2019, Smarandache [10] addressed the problem of allowing the law of composition on X to be either partially defined and partially undefined or totally undefined by introducing the notions of NetroDefined and AntiDefined laws, as well as the notions of NeutroAxiom and AntiAxiom inspired by NL he introduced in 1995. The work of Smarandache in [10] has given birth to the new fields of research called NeutroStructures and AntiStructures. For any classical algebraic law or axiom defined on X, there correspond neutrosophic triads $(<\text{Law}>, <\text{NeutroLaw}>, <\text{AntiLaw}>)$ and $(<\text{Axiom}>, <\text{NeutroAxiom}>, <\text{AntiAxiom}>)$ respectively. In [9], Smarandache studied NeutroAlgebras and AntiAlgebras and in [8], he studied Partial Algebras, Universal Algebras, Effect Algebras and Boole's Partial Algebras and he showed that NeutroAlgebras are generalization of Partial Algebras. In [7], Rezaei and Smarandache studied Neutro-BE-algebras and Anti-BE-algebras and they showed that any classical algebra S with n operations (laws and axioms) where $n \geq 1$ will have $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras. In [2], Agboola et al. studied NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ and in [3], Agboola studied NeutroGroups by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). In addition, he studied NeutroSubgroups, NeutroCyclicGroups, NeutroQuotientGroups and NeutroGroupHomomorphisms. He showed that generally, Lagrange's theorem and fundamental homomorphism theorem of the classical groups do not hold in the class of NeutroGroups studied. In [4], Agboola introduced and studied NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). He presented Several results and examples on NeutroRings, NeutroSubgrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms. He showed that that the fundamental homomorphism of the classical rings holds in the class of NeutroRings considered. Motivated and inspired by the work of Rezaei and Smarandache in [7], Agboola in [1] revisited the NeutroGroups by studying a particular class of NeutroGroups and presented their basic and elementary properties. In the present paper however, the concept of AntiGroups is formally presented. A particular class of AntiGroups is studied with presentation of several examples and basic properties. It is shown that some algebraic properties of the classical groups do not hold in the class of AntiGroups studied. Specifically, it is shown that intersection of two AntiSubgroups is not necessarily an AntiSubgroup and the union of two AntiSubgroups may be an AntiSubgroup. Also, it is shown that Lagranges' theorem and fundamental theorem of homomorphisms of the classical groups do not hold in the class of AntiGroups studied in this paper.

2 Preliminaries

In this section, we will give some definitions and results that will be used later in the paper.

Definition 2.1. [8]

(i) A classical operation is an operation well defined for all the set's elements.

(ii) A NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set.

(iii) An AntiOperation is an operation that is outer defined for all set's elements.

(iv) A classical law/axiom defined on a nonempty set is a law/axiom that is totally true (i.e. true for all set's elements).

(v) A NeutroLaw/NeutroAxiom (or Neutrosophic Law/Neutrosophic Axiom) defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom.

(vi) An AntiLaw/AntiAxiom defined on a nonempty set is a law/axiom that is false for all set's elements.

(vii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom.

(viii) An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

Theorem 2.2. *[7] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.*

Definition 2.3. [Classical group][6]
Let G be a nonempty set and let $* : G \times G \to G$ be a binary operation on G. The couple $(G, *)$ is called a classical group if the following conditions hold:

(G1) $x * y \in G \ \forall x, y \in G$ [closure law].

(G2) $x * (y * z) = (x * y) * z \ \forall x, y, z \in G$ [axiom of associativity].

(G3) There exists $e \in G$ such that $x * e = e * x = x \ \forall x \in G$ [axiom of existence of neutral element].

(G4) There exists $y \in G$ such that $x * y = y * x = e \ \forall x \in G$ [axiom of existence of inverse element] where e is the neutral element of G.

If in addition $\forall x, y \in G$, we have

(G5) $x * y = y * x$, then $(G, *)$ is called an abelian group.

Definition 2.4. [AntiSophication of the law and axioms of the classical group][1]

(AG1) For all the duplets $(x, y) \in G$, $x * y \notin G$ [AntiClosureLaw].

(AG2) For all the triplets $(x, y, z) \in G$, $x*(y*z) \neq (x*y)*z$ [AntiAxiom of associativity (AntiAssociativity)].

(AG3) There does not exist an element $e \in G$ such that $x * e = e * x = x \ \forall x \in G$ [AntiAxiom of existence of neutral element (AntiNeutralElement)].

(AG4) There does not exist $u \in G$ such that $x * u = u * x = e \ \forall x \in G$ [AntiAxiom of existence of inverse element (AntiInverseElement)] where e is an AntiNeutralElement in G.

(AG5) For all the duplets $(x, y) \in G$, $x * y \neq y * x$ [AntiAxiom of commutativity (AntiCommutativity)].

Definition 2.5. [AntiGroup][1]
An AntiGroup AG is an alternative to the classical group G that has at least one AntiLaw or at least one of $\{AG1, AG2, AG3, AG4\}$.

Definition 2.6. [AntiAbelianGroup][1]
An AntiAbelianGroup AG is an alternative to the classical abelian group G that has at least one AntiLaw or at least one of $\{AG1, AG2, AG3, AG4\}$ and $AG5$.

Theorem 2.7. *[1] Let $(G, *)$ be a finite or infinite classical group. Then there are 65 types of AntiGroups.*

Theorem 2.8. *[1] Let $(G, *)$ be a finite or infinite classical abelian group. Then there are 211 types of AntiAbelianGroups.*

Definition 2.9. Let $(AG, *)$ be an AntiGroup. AG is said to be finite of order n if the cardinality of AG is n that is $o(AG) = n$. Otherwise, AG is called an infinite AntiGroup and we write $o(AG) = \infty$.

Since there are many types of AntiGroups, in what follows, AntiGroups will be classified and named type-AG[,] according to which of $AG1 - AG5$ is(are) satisfied. If only $AG1$ is satisfied, the AntiGroup will be called of type-AG[1], type-AG[3,4] if only $AG3$ and $AG4$ are satisfied and so on. AntiGroups of type-AG[1,2,3,4] or of type-AG[1,2,3,4,5] will be called trivial AntiGroups or trivial AntiAbelianGroups respectively.

Example 2.10. Let $AG = \mathbb{Q}_+^*$ be the set of all irrational positive numbers and consider algebraic structure $(AG, .)$ where $''.''$ is the ordinary multiplication of real numbers. Then $(AG, .)$ is an infinite trivial AntiGroup.

Example 2.11. Let $AG = \mathbb{N}$.

(i) Let $*$ be a binary operation on AG defined $\forall x, y \in AG$ by

$$x * y = x + y + xy.$$

Then $(AG, *)$ is a finite AntiGroup of type-AG[3,4].

(ii) Let $*$ be a binary operation on AG defined $\forall x, y \in AG$ by

$$x * y = x + y + 1.$$

Then $(AG, *)$ is a finite AntiGroup of type-AG[3,4].

3 A Study of Finite AntiGroups of Type-AG[4]

In this section, we are going to study a particular class of AntiGroups $(AG, *)$ where $G4$ is totally false for all the elements of AG while $G1, G2, G3$ and $G5$ are either partially true, partially indeterminate or partially false for some elements of AG.

Definition 3.1. Let $(AG, *)$ and $(AH, \circ)$ be AntiGroups of type-AG[4]. The direct product of AG and AH denoted by $AG \times AH$ is defined by

$$AG \times AH = \{(g, h) : g \in AG, h \in AH\}.$$

Proposition 3.2. *Let $(AG, *)$ and $(AH, \circ)$ be AntiGroups of type-AG[4] and let $\otimes$ be a binary operation on $AG \times AH$ defined by*

$$(g, h) \otimes (x, y) = (g * x, h \circ y) \quad \forall\, (g, h), (x, y) \in AG \times AH.$$

Then $(AG \times AH, \otimes)$ is an AntiGroup of type-AG[4].

Proof. The proof follows from the definition of AntiGroups of type-AG[4] and the definition of direct product of AntiGroups of type-AG[4]. □

Proposition 3.3. *Let $(AG, *)$ be an AntiGroup of type-AG[4] and let $g, x, y \in AG$. Then*

(i) $g * x = g * y \;\;\not\Longrightarrow\;\; x = y$.

(ii) $x * g = y * g \;\;\not\Longrightarrow\;\; x = y$.

Proof. Since g^{-1} does not exist and $*$ is NeutroAssociative, the required results follow. □

Proposition 3.4. *Let $(AG, *)$ be an AntiGroup of type-AG[4], $x, y \in AG$ and let $m, n \in \mathbb{N}$. Then*

(i) $x^{m+1} \neq x^m * x$.

(ii) $x^{-m} \neq \left(x^{-1}\right)^m$.

(iii) $x^m * x^{-m} \neq N_e$ *where N_e is a NeutroNeutralElement in AG.*

(iv) $x^m * x^n \neq x^{m+n}$.

(v) $(x^m)^n \neq x^{mn}$.

(vi) $(x * y)^m \neq x^m * y^m$.

Proof. Since x^{-1} does not exist and $*$ is NeutroAssociative, the required results follow. □

Corollary 3.5. *An AntiGroup $(AG, *)$ of type-AG[4] cannot be generated by an element $x \in AG$ and hence cannot by cyclic.*

Definition 3.6. Let $(AG, *)$ be an AntiGroup of type-AG[4]. A nonempty subset AH of AG is called an AntiSubgroup of AG if $(AH, *)$ is also an AntiGroup of the same type as AG. Otherwise, if $(AH, *)$ is an AntiGroup of a type different from the type of AG, then AH is called a QuasiAntiSubgroup of AG.

Definition 3.7. Let $(AG, *)$ be an AntiGroup of type-AG[4] and let AH and AK be AntiSubgroups of AG. The set $A * B$ is defined by

$$A * B = \{x \in AG : x = h * k \quad \text{for some} \quad h \in AH, k \in AK\}.$$

Proposition 3.8. *Let AH, AK and AL be AntiSubgroups of an AntiGroup $(AG, *)$ of type-AG[4]. Then*

(i) $AH * AH \neq AH$.

(ii) $AH * AK \neq AK * AH$.

(iii) $AH * (AK * AL) \neq (AH * AK) * AL$.

Proof. Obvious. □

Definition 3.9. Let $(AG, *)$ be an AntiGroup of type-AG[4] and let $a \in AG$ be a fixed element.

(i) An AntiCenter of AG denoted by $AZ(AG)$ is a set defined by

$$AZ(AG) = \{x \in AG : x * g \neq g * x \quad \forall g \in AG\}.$$

(ii) An AntiCentralizer of $a \in AG$ denoted by AC_a is a set defined by

$$AC_a = \{g \in AG : g * a \neq a * g\}.$$

Example 3.10. Let $\mathbb{U} = \{a, b, c, d, e, f\}$ be a universe of discourse and let $AG = \{a, b, c, e\}$ be a subset of $\mathbb{U}$.

(i) Let $*$ be a binary operation defined on AG as shown in the Cayley table below.

$*$	a	b	c	e
a	d	a	c	e
b	a	d	e	c
c	b	a	f	e
e	a	b	c	f

It is evident from the table that $G1, G2, G3, G5$ are either partially true or partially false with respect to $*$ but $G4$ is totally false for all the elements of AG. Hence $(AG, *)$ is a finite AntiGroup of type-AG[4].

(ii) Let $*$ be a binary operation defined on AG as shown in the Cayley table below.

$*$	e	a	b	c
e	d	a	b	c
a	a	f	c	b
b	b	a	$?$	c
c	c	b	a	$?$

It is evident from the table that $G1, G2, G3, G5$ are either partially true, partially indeterminate or partially false with respect to $*$ but $G4$ is totally false for all the elements of AG. Hence $(AG, *)$ is a finite AntiGroup of type-AG[4].

Example 3.11. Let $(AG, *)$ be the AntiGroup of Example 3.10(i) and let $AH_1 = \{a, b, e\}$ and $AH_2 = \{b, c, e\}$ be two subsets of AG. Let $*$ be defined on AH_1 and AH_2 as shown in the Cayley tables below:

AH_1 :

$*$	a	b	e
a	d	a	e
b	a	d	c
e	a	b	f

AH_2 :

$*$	b	c	e
b	d	e	c
c	a	f	e
e	b	c	f

It can easily be seen from the tables that AH_1 is an AntiSubgroup of AG while AH_2 is a QuasiAntiSubgroup of AG. It is noted that Lagranges' theorem does not hold. It is also noted that:

$$\begin{aligned} AH_1 \cup AH_2 &= \{a, b, c, e\} = AG, \\ NH_1 \cap NH_2 &= \{b, e\}, \end{aligned}$$

from which it is deduced that $NH_1 \cup NH_2$ is an AntiSubgroup of AG but $AH_1 \cap AH_2$ is a QuasiAntiSubgroup of AG as it is evident in the Cayley table below:

$NH_1 \cap NH_2$:

$*$	b	e
b	d	c
e	b	f

$AZ(AG) = \{a, b, c, e\} = AG$ is an AntiSubgroup of AG. Also, $AC_a = AC_b = AC_c = AC_e = \{a, b, c, e\} = AG$ are AntiSubgroups of AG.

Example 3.12. Let $(AG, *)$ be the AntiGroup of Example 3.10(ii) and let $AH_1 = \{e, a, b\}$ and $AH_2 = \{e, b, c\}$ be two subsets of AG. Let $*$ be defined on AH_1 and AH_2 as shown in the Cayley tables below:

AH_1 :

$*$	e	a	b
e	d	a	b
a	a	f	c
b	b	a	$?$

AH_2 :

$*$	e	b	c
e	d	b	c
b	b	$?$	c
c	c	a	$?$

It can easily be seen from the tables that AH_1 and AH_2 are AntiSubgroups of AG. It is noted that Lagranges' theorem does not hold. It is also noted that:

$$\begin{aligned} AH_1 \cup AH_2 &= \{e, a, b, c\} = AG, \\ NH_1 \cap NH_2 &= \{e, b\}, \end{aligned}$$

from which it is deduced that $NH_1 \cup NH_2$ is an AntiSubgroup of AG and $AH_1 \cap AH_2$ is an AntiSubgroup of AG as it is evident in the Cayley table below:

$NH_1 \cap NH_2$:

$*$	e	b
e	d	f
b	b	$?$

$AZ(AG) = \{a, b, c\}$ is a QuasiAntiSubgroup of AG. Also, $AC_a = AC_b = \{a, b, c\}$, $AC_c = \{b, c\}$ are QuasiAntiSubgroups of AG and $AC_e = \{\} = \emptyset$ is neither an AntiSubgroup nor a QuasiAntiSubgroup of AG.

Example 3.13. (i) Let $AG = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and let $\oplus$ be a binary operation on AG as defined in the Cayley table.

$\oplus$	0	1	2	3
0	0 or 1	1	2	3
1	1	2	3	0 or 2
2	2	3	0 or 3	1
3	3	?	1	2

Then $(AG, \oplus)$ is a finite AntiGroup of type-AG[4] and $AH = \{0, 1, 2\}$ is an AntiSubgroup of AG.

(ii) Let $AG = \{1, 2, 3, 4\} \subseteq \mathbb{Z}_5$ and let $\otimes$ be a binary operation on AG as defined in the Cayley table.

$\otimes$	1	2	3	4
1	?	2	3	4
2	2	4	0	3
3	3	?	4	2
4	4	3	2	0

Then $(AG, \otimes)$ is a finite AntiGroup of type-AG[4] and $AH = \{1, 2, 3\}$ is an AntiSubgroup of AG.

Definition 3.14. Let AH be an AntiSubgroup of the AntiGroup $(AG, *)$ of type-AG[4] and let $x \in AG$.

(i) $x * AH$ the left coset of AH in AG is defined by

$$x * AH = \{x * h : h \in AH\}.$$

(ii) $AH * x$ the right coset of AH in AG is defined by

$$AH * x = \{h * x : h \in AH\}.$$

(iii) AH is called a normal AntiSubgroup if $x * AH = AH * x$ for at least one $x \in AG$.

(iv) The number of distinct left or right cosets of AH in AG is called the index of AH in AG and it is denoted by $[AG : AH]$.

(vi) The set of all distinct left cosets of AH in AG denoted by $(AG/AH)_L$ is defined by

$$(AG/AH)_L = \{x * AH : x \in AG\}.$$

(vii) The set of all distinct right cosets of AH in AG denoted by $(AG/AH)_R$ is defined by

$$(AG/AH)_R = \{AH * x : x \in AG\}.$$

Suppose that AG/AH is the set of all distinct left cosets of AH in AG and suppose that $\odot$ is a binary operation on AG/AH defined by

$$(x * AH) \odot (y * AH) = (x * y) * AH \quad \forall\, x * AH, y * AH \in AG/AH.$$

If the couple $(AG/AH, \odot)$ is an AntiGroup of type-AG[4], then AG/AH is called an AntiQuotientGroup of AG factored by AH.

Lemma 3.15. *Let AH be an AntiSubgroup of the AntiGroup $(AG, *)$ of type-AG[4] and let $e \in AG$ be a NeutroNeutralElement. Then globally,*

$$e * AH \neq AH.$$

Example 3.16. Let $(AG, \oplus)$ be an AntiGroup of Example 3.13 (i) and let $AH = \{0, 1, 2\}$ be its AntiSubgroup. The left and right cosets of AH in AG are:

$$\begin{aligned} 0 \oplus AH &= \{0,1,2\} \text{ or } \{1,2\} = AH \oplus 0, \\ 1 \oplus AH &= \{1,2,3\}, = AH \oplus 1 \\ 2 \oplus AH &= \{0,2,3\} \text{ or } \{2,3\} = AH \oplus 2, \\ 3 \oplus AH &= \{1,3,?\} = AH \oplus 3, \\ \therefore \quad AG/AH &= \{0 \oplus AH, 1 \oplus AH, 2 \oplus AH, 3 \oplus AH\}. \end{aligned}$$

It is noted that AH is a normal AntiSubgroup of AG and distinct left and right cosets of AH do not partition AG.

Now consider the Cayley table below.

$\bigoplus$	$0 \oplus AH$	$1 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$
$0 \oplus AH$	$0 \oplus AH$ or $1 \oplus AH$	$1 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$
$1 \oplus AH$	$1 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$	$0 \oplus AH$ or $2 \oplus AH$
$2 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$	$0 \oplus AH$ or $3 \oplus AH$	$1 \oplus AH$
$3 \oplus AH$	$3 \oplus AH$	?	$1 \oplus AH$	$2 \oplus AH$

It is evident from the table that $(AG/AH, \bigoplus)$ is an AntiGroup of type-AG[4].

Example 3.17. Let $(AG, \otimes)$ be an AntiGroup of Example 3.13 (ii) and let $AH = \{1, 2, 3\}$ be its AntiSubgroup. The left cosets of AH in AG are:

$$\begin{aligned} 1 \otimes AH &= \{?,2,3\} = AH \otimes 1, \\ 2 \otimes AH &= \{0,2,4\} = AH \otimes 2, \\ 3 \otimes AH &= \{3,4,?\} = AH \otimes 3, \\ 4 \otimes AH &= \{2,3,4\} = AH \otimes 4, \\ \therefore \quad AG/AH &= \{1 \otimes AH, 2 \otimes AH, 3 \otimes AH, 4 \otimes AH\}. \end{aligned}$$

It is noted that AH is a normal AntiSubgroup of AG and distinct left and right cosets of AH do not partition AG.

Now consider the Cayley table below.

$\bigotimes$	$1 \otimes AH$	$2 \otimes AH$	$3 \otimes AH$	$4 \otimes AH$
$1 \otimes AH$	?	$2 \otimes AH$	$3 \otimes AH$	$4 \otimes AH$
$2 \otimes AH$	$2 \otimes AH$	$4 \otimes AH$	$0 \otimes AH$	$3 \otimes AH$
$3 \otimes AH$	$3 \otimes AH$	?	$4 \otimes AH$	$2 \otimes AH$
$4 \otimes AH$	$4 \otimes AH$	$3 \otimes AH$	$2 \otimes AH$	$0 \otimes AH$

It is evident from the table that $(AG/AH, \bigotimes)$ is an AntiGroup of type-AG[4].

Proposition 3.18. *Let AH be a normal AntiSubgroup of an AntiGroup $(AG, *)$ of type-AG[4] and let AG/AH be the set of distinct left cosets of AH in AG. For $x * AH, y * AH \in AG/AH$ with $x, y \in AG$, let $\odot$ be a binary operation defined on AG/AH by*

$$(x * AH) \odot (y * AH) = (x * y) * AH \quad \forall\, x, y \in AG.$$

Then, $(AG/AH, \odot)$ is an AntiGroup of type-AG[4].

Proof. Suppose that AH is a normal AntiSubgroup of an AntiGroup $(AG, *)$ of type-AG[4] and suppose that the composition of elements in AG/AH is given by $(x * AH) \odot (y * AH) = (x * y) * AH \quad \forall\, x, y \in AG$. Then there exist some duplets $(x, y), (u, v), (p, q) \in AG$ such that $x * y \in AG$ (inner-defined) and [$u * v =$ indeterminate or $p * q \notin AG$ (outer-defined/falsehood)]. Hence, $\odot$ satisfies the NeutroClosureLaw. Next, there exist some triplets $(x, y, z), (p, q, r), (u, v, w) \in AG$ such that $x * (y * z) = (x * y) * z$ (inner-defined) and [[$p * (q * r)$]or[$(p * q) * r$] $=$ indeterminate or $u * (v * w) \neq (u * v) * w$ (outer-defined/falsehood)]. This again shows that $\odot$ satisfies the NeutroAssociativityAxiom. Also, there exists an element $e \in AG$ such that $x * e = e * x = x$ (inner-defined) and [[$x * e$]or[$e * x$] $=$ indeterminate or $x * e \neq x \neq e * x$ (outer-defined/falsehood)] for at least one $x \in AG$. This shows the existence of NeutroNeutralElement in AG and hence there exists a NeutroNeutralElement $e * AH \in AG/AH$. Again for all $x \in AG$, there does not exist $u \in AG$ such that $x * u = u * x = e$. This is an AntiAxiom of existence of inverse element in AG and consequently, no element $x * AH \in AG/AH$ has an inverse. Lastly, there exist some duplets $(x, y), (u, v), (p, q) \in AG$ such that $x * y = y * x$ (inner-defined) and [[$u * v$]or[$v * u$] $=$ indeterminate or $p * q \neq q * p$ (outer-defined/falsehood)]. This shows that $\odot$ satisfies the NeutroCommutativityAxiom. Hence,$(AG/AH, \odot)$ is an AntiGroup of type-AG[4]. □

Definition 3.19. Let $(AG, *)$ and $(AH, \circ)$ be any two AntiGroups of type-AG[4]. The mapping $\phi : AG \to AH$ is called an AntiGroupHomomorphism if ϕ does not preserve the binary operations $*$ and $\circ$ that is for all the duplet $(x, y) \in AG$, we have

$$\phi(x * y) \neq \phi(x) \circ \phi(y).$$

The kernel of ϕ denoted by $Ker\phi$ is defined by

$$Ker\phi = \{x : \phi(x) = e_{AH} \text{ for at least one } e_{AH} \in AH\}$$

where e_{AH} is a NeutroNeutralElement in AH.

The image of ϕ denoted by $Im\phi$ is defined by

$$Im\phi = \{y \in AH : y = \phi(x) \text{ for some } x \in AG\}.$$

If in addition ϕ is an AntiBijection, then ϕ is called an AntiGroupIsomorphism. AntiGroupEpimorphism, AntiGroupMonomorphism, AntiGroupEndomorphism, and AntiGroupAutomorphism are defined similarly.

Example 3.20. (i) Let $(AG, \oplus)$ be the AntiGroup of Example 3.13 (i) and let $\phi : AG \to AG$ be a mapping defined by

$$\phi(x) = 2 \oplus x \quad \forall\, x \in AG.$$

Then

$$\begin{aligned} \phi(0) &= 2, \\ \phi(1) &= 3, \\ \phi(2) &= 0 \text{ or } 3, \\ \phi(3) &= 1, \end{aligned}$$

from which we obtain that $\phi(x \oplus y) \neq \phi(x) \oplus y$ for all $x, y \in AG$. Accordingly, ϕ is an AntiGroupHomomorphism. $Im\phi = \{1, 2, 3\}$ which is an AntiSubgroup of AG. $Ker\phi = \{\} = \emptyset$.

(ii) Let $(AG, \otimes)$ be the AntiGroup of Example 3.13 (ii) and let $\psi : AG \to AG$ be a mapping defined by

$$\psi(x) = x \otimes 4 \quad \forall\, x \in AG.$$

Then

$$\begin{aligned} \psi(1) &= 4, \\ \psi(2) &= 3, \\ \psi(3) &= 2, \\ \psi(4) &= 0, \end{aligned}$$

from which we obtain that $\psi(x \otimes y) \neq \psi(x) \otimes y$ for all $x, y \in AG$. Accordingly, ψ is an AntiGroupHomomorphism. $Im\phi = \{0, 2, 3, 4\}$ which is not an AntiSubgroup of AG. $Ker\phi = \{\} = \emptyset$.

Example 3.21. (i) Let $(AG, *)$ be the AntiGroup of Example 3.10 (i) and let $\phi : AG \times AG \to AG$ be a projection defined by

$$\phi((x, y)) = x \quad \forall\, x, y \in AG.$$

Then ϕ is not an AntiGroupHomomorphism because $\phi((a, b) \otimes (b, c)) = \phi((a, b)) * \phi((b, c)) = a$. However, $Im\phi = \{a, b, c, e\} = AG$.

(ii) Let $(AG, *)$ be the AntiGroup of Example 3.10 (ii) and let $\psi : AG \times AG \to AG$ be a projection defined by

$$\psi((x, y)) = y \quad \forall\, x, y \in AG.$$

Then ψ is not an AntiGroupHomomorphism because $\psi((a, b) \otimes (b, c)) = \psi((a, b)) * \psi((b, c)) = c$. However, $Im\psi = \{a, b, c, e\} = AG$.

Example 3.22. (i) Let $(AG/AH, \bigoplus)$ be the AntiQuotientGroup of Example 3.16 and let $\phi : AG \to AG/AH$ be a mapping defined by

$$\phi(x) = x \oplus AH \quad \forall\, x \in AG.$$

Then

$$\begin{aligned} \phi(0) &= 0 \oplus AH, \\ \phi(1) &= 1 \oplus AH, \\ \phi(2) &= 2 \oplus AH, \\ \phi(3) &= 3 \oplus AH, \end{aligned}$$

from which we obtain

$$\phi(1 \oplus 2) = \phi(1) \bigoplus \phi(2) = 3 \oplus AH.$$

This shows that ϕ is not an AntiGroupHomomorphism.

(ii) Let $(AG/AH, \bigotimes)$ be the AntiQuotientGroup of Example 3.17 and let $\psi : AG \to AG/AH$ be a mapping defined by

$$\psi(x) = x \otimes AH \quad \forall\, x \in AG.$$

Then

$$\begin{aligned} \psi(1) &= 1 \otimes AH, \\ \psi(2) &= 2 \otimes AH, \\ \psi(3) &= 3 \otimes AH, \\ \psi(4) &= 4 \otimes AH, \end{aligned}$$

from which we obtain

$$\psi(2 \otimes 4) = \psi(2) \bigotimes \psi(4) = 3 \otimes AH.$$

This shows that ψ is not an AntiGroupHomomorphism.

Remark 3.23. The fundamental theorem of homomorphisms of the classical groups cannot hold in the class of AntiGroups of type-AG[4] as demonstrated in Examples 3.22 (i) and (ii).

4 Conclusion

The notion of AntiGroups was formally presented in this paper. A particular class of AntiGroups of type-AG[4] was studied. In AntiGroups of type-AG[4], the existence of an inverse element was taking to be totally false for all the elements while the closure law, the existence of identity element, the axioms of associativity and commutativity were taking to be either partially true, partially indeterminate or partially false for some elements. It was shown that some algebraic properties of the classical groups do not hold in the class of AntiGroups of type-AG[4]. Specifically, it was shown that intersection of two AntiSubgroups is not necessarily an AntiSubgroup and the union of two AntiSubgroups may be an AntiSubgroup. Also, it was shown that distinct left(right)cosets of AntiSubgroups of AntiGroups of type-AG[4] do not partition the AntiGroups; and that Lagranges' theorem and fundamental theorem of homomorphisms of the classical groups do not hold in the class of AntiGroups of type-AG[4]. More classes of AntiGroups will be studied in our future papers.

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n-Cyclic Refined Neutrosophic Algebraic Systems of Sub-Indeterminacies, An Application to Rings and Modules

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Abstract

Refining the indeterminate I into many levels of indeterminacy is a way to explore many neutrosophic algebraic structures.This paper introduces the concept of n-cyclic refined algebraic system of sub-indeterminacies as a new way to refine a neutrosophic indeterminate I. This idea will be used to introduce the notion of n-cyclic refined neutrosophic ring and to study its AH-substructures. Also, this work presents the concept of n-cyclic refined neutrosophic modules with many related structures.

Keywords: n-cyclic refined neutrosophic system, n-cyclic refined neutrosophic ring, n-cyclic refined neutrosophic AH-substructures, n-cyclic refined neutrosophic modules, n-cyclic refined neutrosophic homomorphisms.

1.Introduction

Neutrosophy is one of the most important concepts in philosophy, while its derivative the notion of neutrosophic set has been used widely to define new algebraic structures and topologies such as neutrosophic groups, rings, spaces, and modules. See [1,3,4,5,7,8,9,13]. Recently, many generalizations were defined such as refined neutrosophic set [10]. It has been used in the study of refined neutrosophic rings [6], refined neutrosophic modules and spaces [12,14,15,16].

In [20], F. Smarandache refined the literal indeterminacy (I) into subindeterminacies I_1, I_2, …, I_n, and defined the multiplication law of subindeterminacies.

In [18], Smarandache e tal. presented the concept of n-refined neutrosophic set by splitting the indeterminacy I into n-degrees of indeterminacy $I_1, \dots, I_n$. By defining an algebraic operation between sub-indeterminacies, Smarandache and Abobala introduced n-refined neutrosophic rings, n-refined neutrosophic vector spaces and modules, AH-structures in n-refined neutrosophic vector spaces [2,17,18,19].

The algebraic operation between sub-indeterminacies was defined as follows:

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$I_i \cdot I_j = I_{\min(i,j)}$. Through this paper, we will define another closed algebraic operation between sub-indeterminacies, which allows us to build a new refined neutrosophic system and new n-refined neutrosophic algebraic structures.

We say that $I_1, \dots, I_n$ is an n-cyclic refining system of indeterminacies if it has the following binary operation $I_i . I_j = I_{(i+j mod n)}$. For example if $n = 4$, we have $\{I_1, I_2, I_3, I_4\}$ as 4-cyclic refining system with: $I_2 I_3 = I_1, I_3 I_3 = I_2, I_3 I_1 = I_4, I_4 I_3 = I_3$.

We remark that n-cyclic refining system $I_1, \dots, I_n$ has a cyclic group structure with order n.

Another important remark is about that the elements of any n-refined neutrosophic ring are exactly equal to the elements of n-cyclic refined neutrosophic ring; but the products between elements are totally different according to the definition of multiplication between sub-indeterminacies, which means that we get a new kind of refined neutrosophic rings. Hence, there is a lot of similarity between this work and our paper [18], that is because we use the same symbols and ordering of phrases.

For easy expression, we assume that $aI_0 = a$ and $I_0 I_j = I_j$. For all possiple values of j.

2. Preliminaries

Definition 2.1: [18]

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n subindeterminacies. We define R_n(I)=$\{a_0 + a_1 I + \dots + a_n I_n ;\ a_i \in R\}$ to be n-refined neutrosophic ring.

Definition 2.2: [18]

(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\{a_0 + a_1 I_1 + \dots + a_n I_n : a_i \in P_i\}$, where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all i. AHS-subring is defined by the condition $P_i = P_j$ for all i , j.

(b) P is an AH-ideal ifP_i is an two sides ideal of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i , j.

(c) The AH-ideal P is said to be null if $P_i = R or P_i = \{0\}$ for all i.

Definition 2.3: [8]

Let (M, +, .) be a module over the ring R then (M(I) ,+, .) is called a weak neutrosophic module over the ring R, and it is called a strong neutrosophic module if it is a module over the neutrosophic ring R(I).

Elements of M(I) have the form $\boldsymbol{x + yI; x , y \in M}$, i.e M(I) can be written as $\boldsymbol{M(I) = M + MI}$.

Definition 2.4: [8]

Let M(I) be a strong neutrosophic module over the neutrosophic ring R(I) and W(I) be a non empty subset of M(I), then W(I) is called a strong neutrosophic submodule if W(I) itself is a strong neutrosophic module.

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Definition 2.5: [8]

Let U(I) and W(I) be two strong neutrosophic submodules of M(I) and let $f: U(I) \to W(I)$, we say that f is a neutrosophic module homomorphism if

(a) $f(I) = I$.

(b) f is a module homomorphism.

3. n-Cyclic refined neutrosophic rings

Definition 3.1:

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)=$\{a_0 + a_1 I + \cdots + a_n I_n \; ; \; a_i \in R\}$ to be n-cyclic refined neutrosophic ring.

Operations on R_n(I) are defined as:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i \, , \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} \left(x_i \times y_j\right) I_i I_j = \sum_{i,j=0}^{n} \left(x_i \times y_j\right) I_{(i+j mod n)}$$

Where × is the multiplication on the ring R.

It is obvious that R_n(I) is a ring in the algebraic ordinary concept.

Definition 3.2:

Let R_n(I) be an n-cyclic refined neutrosophic ring, it is called commutative if $xy = yx$ for all x , y ∈ R_n(I). If there is 1∈ R_n(I) such that $1.x = x.1 = x$, then it is called an n-cyclic refined neutrosophic ring with unity.

Theorem 3.3:

Let R_n(I) be an n-cyclic refined neutrosophic ring. Then

(a) R is commutative if and only if R_n(I) is commutative,

(b) R has unity if and only if R_n(I) has unity,

(c) R_n(I) = $\sum_{i=0}^{n} R I_i$ = $\{\sum_{i=0}^{n} x_i I_i : x_i \in R\}$.

Proof:

It is trivial, and similar to the classical case.

Definition 3.4:

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(a) Let R_n(I) be an n- cyclic refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i = \{a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i\}$ where P_i is a subset of R, we define P to be an n-cyclic AH-subring if P_i is a subring of R for all i, n-cyclic AHS-subring is defined by the condition $P_i = P_j$ for all i , j.

(b) P is an n-cyclic AH-ideal ifP_i is a two-sided ideal of R for all i, the n-cyclic AHS-ideal is defined by the condition $P_i = P_j$ for all i , j.

(c) The n-cyclic AH-ideal P is said to be null if $P_i = R or P_i = \{0\}$ for all i.

Ideals and subrings by classical meaning need not to be defined, that is because they are well defined in classical studies.

Theorem 3.5:

Let R_n(I) be an n-cyclic refined neutrosophic ring and P is an AH-ideal, (P,+) is an abelian group with $k \leq n$ and r.p$\in P$ for all p $\in P$ and r $\in R$.

Proof :

Since P_i is abelian subgroup of $(R, +)$ and $r.x \in P_i$ for all $r \in R , x \in P_i$, the required result follows.

Definition 3.6:

Let R_n(I) be an n-cyclic refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$, Q = $\sum_{i=0}^{n} Q_i I_i$ be two AH-ideals, we define:

P+Q = $\sum_{i=0}^{n}(P_i + Q_i)I_i$, P∩Q = $\sum_{i=0}^{n}(P_i \cap Q_i)I_i$.

Theorem 3.7:

Let R_n(I) be any n-cyclic refined neutrosophic ring, P = $\sum_{i=0}^{n} P_i I_i$, Q = $\sum_{i=0}^{n} Q_i I_i$ be two n-cyclic AH- ideals, then P+Q, P∩Q are n-cyclic AH-ideals. If P, Q are n-cyclic AHS-ideals ,then P+Q, P∩Q are n-cyclic AHS-ideals.

Proof :

The proof is similar to the case of n-refined neutrosophic rings in [18].

Definition 3.8:

Let R_n(I) be an n-cyclic refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$be an n-cyclic AH-ideal, the AH- radical of P can be defined as $AH - rad(P) = \sum_{i=0}^{n}(\sqrt{P_i})I_i$.

Theorem 3.9:

The AH-radical of an AH-ideal is an AH-ideal in any n-cyclic refined neutrosophic ring.

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Proof :

The proof is similar to the case of n-refined neutrosophic ring in [18].

Definition 3.10:

Let R_n(I) be an n-cyclic refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ be an AH-ideal, we define the n-cyclic AH-factor R(I)/P = $\sum_{i=0}^{n}(R/P_i)I_i = \sum_{i=0}^{n}(x_i + P_i)I_i \,; x_i \in R$.

Theorem 3.11:

Let R_n(I) be an n-cyclic refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ be an n-cyclic AH-ideal:

R_n(I)/P is a ring with the following two binary operations

$\sum_{i=0}^{n}(x_i + P_i)I_i + \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i + y_i + P_i)I_i$,

$\sum_{i=0}^{n}(x_i + P_i)I_i \times \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i \times y_i + P_i)I_i$.

Proof :

The proof is similar to Theorem 3.12 in [18].

Definition 3.12:

(a) Let R_n(I), T_n(I) be two n-cyclic refined neutrosophic rings respectively, and $f_R: R \to T$ be a ring homomorphism. We define n-cyclic refined neutrosophic AHS-homomorphism as follows:

f: R_n(I)→ T_n(I); $f(\sum_{i=0}^{n} x_i I_i) = \sum_{i=0}^{n} f_R(x_i) I_i$.

(b) f is an n-cyclic refined neutrosophic AHS-isomorphism if it is a bijective n-cyclic refined neutrosophic AHS-homomorphism.

(c) AH-Ker f = $\sum_{i=0}^{n} Ker(f_R) I_i$= $\{\sum_{i=0}^{n} x_i I_i \,;\ x_i \in Ker f_R\}$.

Theorem 3.13:

Let R_n(I), T_n(I) be two n-cyclic refined neutrosophic rings respectively and f be an n-cyclic refined neutrosophic AHS-homomorphism f: R_n(I)→ T_n(I). Then

(a) If P = $\sum_{i=0}^{n} P_i I_i$ is an n-cyclic AH- subring of R_n(I), then f(P) is an n-cyclic AH- subring of T_n(I),

(b) If P = $\sum_{i=0}^{n} P_i I_i$ is an n-cyclic AHS- subring of R_n(I), then f(P) is an n-cyclic AHS- subring of T_n(I),

(c) If P = $\sum_{i=0}^{n} P_i I_i$ is an n-cyclic AH-ideal of R_n(I), then f(P) is an n-cyclic AH-ideal of f(R_n(I)),

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(d) P = $\sum_{i=0}^{n} P_i I_i$ is an n-cyclic AHS-ideal of R_n(I), then f(P) is an n-cyclic AHS-ideal of f(R_n(I)),

(e) $R_n(\mathrm{I})/AH - Ker(f)\, isAHS - isomorphictof\big(R(I)\big),$

(f) The inverse image of an n-cyclic AH-ideal P in T_n(I) is an n-cyclic AH-ideal in R(I).

Proof :

(a) Since $f(P_i)$ is a subring of T, then f(P) is an n-cyclic AH- subring of T_n(I).

(b) Holds by a similar way to (a).

(c) Since $f(P_i)$ is an ideal of f(R), then f(P) is an n-cyclic AH- ideal of f(R(I)).

(d) It is similar to (c).

(e) We have $R/Ker(f_R) \cong f(R)$, by definition of n-cyclic AH-factor and $AH - Ker(f)$ we find that $R(I)/P \cong f(R(I))$.

(f) It is similar to the classical case.

Definition 3.14:

(a) Let R(I) be a commutative n-cyclic refined neutrosophic ring, and P = $\sum_{i=0}^{n} P_i I_i$ be an n-cyclic AH- ideal, we define P to be a weak prime n-cyclic AH-ideal if P_i is a prime ideal of R for all i .

(b) P is called a weak maximal n-cyclic AH-ideal if P_i is a maximal ideal of R for all i.

(c) P is called a weak principal n-cyclic AH-ideal if P_i is a principal ideal of R for all i.

Theorem 3.15:

Let R_n(I), T_n(I) be two commutative n-cyclic refined neutrosophic rings with an n-cyclic refined neutrosophic AHS-homomorphism f: R_n(I)→ T_n(I):

(a) If P = $\sum_{i=0}^{n} P_i I_i$ is an n-cyclic AHS- ideal of R_n(I) and Ker(f_R) $\leq P_i \neq R_n$(I):

(a) P is a weak prime n-cyclic AHS-ideal if and only if f(P) is a weak prime AHS-ideal in f(R_n(I)).

(b) P is a weak maximal n-cyclic AHS-ideal if and only if f(P) is a weak maximal n-cyclic AHS-ideal in f(R_n(I)).

(c) If Q = $\sum_{i=0}^{n} Q_i I_i$ is an n-cyclic AHS-ideal of T_n(I), then it is a weak prime n-cyclic AHS-ideal if and only if $f^{-1}(Q)$ is a weak prime inR_n(I).

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(d)if Q = $\sum_{i=0}^{n} Q_i I_i$ is an n-cyclic AHS-ideal of T_n(I), then it is a weak maximal AHS-ideal if and only if $f^{-1}(Q)$ is a weak maximal in R_n(I).

Proof :

Proof is similar to Theorem 16.3 in[18].

Example 3.16:

Let $R = Z\ be$ the ring of integers, $T = Z_6$ be the ring of integers modulo 6 with multiplication and addition modulo 6, we have:

(a) $f_R: R \to T; f(x) = xmod6$ is a ring homomorphism, $ker(f_R) = 6Z,$ the corresponding 4-cyclic AHS-homomorphism between R_4(I), T_4(I) is:

$f: R_4(\mathrm{I}) \to T_4(\mathrm{I}); f(a + bI_1 + cI_2 + dI_3 + eI_4) = (amod6) + (bmod6)I_1 + (cmod6)I_2 + (dmod6)I_3 + (emod6)I_4;\ a, b, c, d, e \in Z.$

(b) $P =< 2 >, Q =< 3 >$are two prime and maximal and principal ideals in R,

M= $P + PI_1 + QI_2 + QI_3 + PI_4 = \{(2a + 2bI_1 + 3cI_2 + 3dI_3 + 2eI_4; a, b, c, d, e \in Z\}$ is a weak prime/ maximal 4-cyclic AH-ideal of R_4(I).

(c) $Ker(f_R) = 6Z \le P, Q, f_R(P) = \{0,2,4\}, f_R(Q) = \{0,3\},$

$f(M) = f(P) + f(P)I_1 + f(Q)I_2 + f(Q)I_3 + f(P)I_4$ which is a weak maximal/ prime/principal 4-cyclic AH-ideal of T_4(I).

(d) $AH - Ker(f) = 6Z + 6ZI_1 + 6ZI_2 + 6ZI_3 + 6ZI_4$ which is an 4-cyclic AHS-ideal of $R_4(I)$.

(e) $R_4(I)/AH - Kerf = R/6Z + R/6Z\, I_1 + R/6Z\, I_2 + R/6Z\, I_3 + R/6Z\, I_4$which is AHS-isomorphic to $f(R_4(I)) = T_4(I)$, since $R/6Z \cong T$.

Example 3.17:

Let $R = Z_8$ be a ring with addition and multiplication modulo 8.

(a) 3-cyclic refined neutrosophic ring related with R is Z_{8_3}(I)={a+$bI_1 + cI_2 + dI_3; a, b, c, d \in Z_8$}.

(b) P={0,4} is an ideal of R, $\sqrt{P}$={0,2,4,6},$M = P + PI_1 + PI_2 + PI_3$ is an 3-cyclic AHS-ideal of Z_{8_3}(I),

$AH - Rad(M) = \sqrt{P} + \sqrt{P}I_1 + \sqrt{P}I_2 + \sqrt{P}I_3$ which is a 3-cyclic AHS-ideal of Z_{8_3}(I).

Example 3.18:

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Let R=Z_2 be the ring of integers modulo 2, let $n = 3$. The corresponding 3-cyclic refined neutrosophic ring is

Z_{2_3}(I)={$0,1,I_1, I_2, I_3, 1 + I_1, 1 + I_2, 1 + I_3, I_1 + I_2, I_1 + I_3, I_1 + I_2 + I_3, I_2 + I_3, 1 + I_1 + I_2 + I_3, 1 + I_2 + I_3, 1 + I_1 + I_3, 1 + I_1 + I_2$}.

We can remark that elements in the n-cyclic refined neutrosophic ring are exactly equal to elements in the corresponding n-refined neutrosophic ring; but the products of them are different.

4. n-Cyclic refined neutrosophic polynomial rings

Definition 4.1:

Let R_n(I) be a commutative n-cyclic refined neutrosophic ring and $P: R_n$(I) $\rightarrow R_n$(I) is a function defined as $P(x) = \sum_{i=0}^{m} a_i x^i$ such that$a_i \in R_n$(I), we call P an n-cyclic refined neutrosophic polynomial on R_n(I).

We denote byR_n(I)[x] to be the ring of n-cyclic refined neutrosophic polynomials overR_n(I).

Since R_n(I) is a classical ring, then R_n(I)[x] is a classical ring.

Theorem 4.2:

Let $R_n(I)$ be a commutative n-cyclic refined neutrosophic ring. Then R_n(I)[x] = $\sum_{i=0}^{n}$ R$[x]I_i$.

Proof :

Let P(x) = $\sum_{i=0}^{n} P_i(x)I^i \in \sum_{i=0}^{n}$ R$[x]I^i$,by rearranging the previous sum we can write it as P(x) = $\sum_{i=0}^{m} a_i x^i \in R_n(I)[x]$.

Conversely, if P(x) = $\sum_{i=0}^{n} a_i x^i \in R_n(I)[x]$, then we can write it as

P(x) = $\sum_{i=0}^{n} P_i(x)I_i \in \sum_{i=0}^{n} R[x]I_i$, by the previous argument we find the proof.

Example 4.3:

Let $Z_3(I)$ be a 3-cyclic refined neutrosophic ring and P(x) = I_1 + (2+I_1) x + (I_1+I_3)x^2 a polynomial over $Z_{3_n}(I)$, then we can write P(x) = 2x + I_1 (1 + x + x^2) + $I_2 x^2$.

It is obvious that R_n(I)$\leq R_n$(I)$[x]$.

Definition 4.4:

Let P(x) = $\sum_{i=0}^{n} P_i(x)I^i$ a neutrosophic polynomial over R_n(I) we define the degree of P by deg P = max(deg P_i).

5. n-Cyclic refined neutrosophic modules

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Definition 5.1 :

Let (M,+,.) be a module over the ring R, we say that $M_n(I) = M + MI_1 + \cdots + MI_n = \{x_0 + x_1I_1 + \cdots + x_nI_n;\ x_i \in M\}$ is a weak n-cyclic refined neutrosophic module over the ring R. Elements of $M_n(I)$ are called n-cyclic refined neutrosophic vectors, elements of R are called scalars.

If we take scalars from the n- cyclic refined neutrosophic ring $R_n(I)$, we say that $M_n(I)$ is a strong n-cyclic refined neutrosophic module over the n-cyclic refined neutrosophic ring $R_n(I)$. Elements of $M_n(I)$ are called n-refined neutrosophic scalars.

Remark 5.2:

Addition on $M_n(I)$ is defined as:

$$\sum_{i=0}^{n} a_iI_i + \sum_{i=0}^{n} b_iI_i = \sum_{i=0}^{n} (a_i + b_i)I_i.$$

Multiplication by a scalar $m \in R$ is defined as:

$m.\sum_{i=0}^{n} a_iI_i = \sum_{i=0}^{n}(m.a_i)I_i.$

Multiplication by an n-cyclic refined neutrosophic scalar $m = \sum_{i=0}^{n} m_iI_i \in R_n(I)$ is defined as:

$\sum_{i=0}^{n} m_iI_i\ .\sum_{i=0}^{n} a_iI_i = \sum_{i=0}^{n}(m_i.a_i)I_iI_j.$

Where $\mathrm{a_i \in M, m_i \in R, I_iI_j = I_{(i+jmodn)}}$.

Theorem 5.3 :

Let (M,+,.) be a module over the ring R. Then a weak n-cyclic refined neutrosophic module $M_n(I)$ is a module over the ring R. A strong n-cyclic refined neutrosophic module is a module over the n-cyclic refined neutrosophic ring R_n(I).

Proof:

It is similar to the classical case.

Example 5.4:

Let $M = Z_2$ be the finite module of integers modulo 2 over itself, we have:

(a) The corresponding weak 2-cyclic refined neutrosophic module over the ring Z_2 is

$M_n(I) = \{0,1, I_1, I_2, I_1 + I_2, 1 + I_1 + I_2, 1 + I_1, 1 + I_2\}.$

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Definition 5.5:

Let $M_n(I)$ be a weak n-cyclic refined neutrosophic module over the ring R, a nonempty subset $W_n(I)$ is called a weak n-cyclic refined neutrosophic module of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Definition 5.6:

Let $M_n(I)$ be a strong n-cyclic refined neutrosophic module over the n-cyclic refined neutrosophic ring $R_n(I)$, a nonempty subset $W_n(I)$ is called a strong n-cyclic refined neutrosophic submodule of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Theorem 5.7:

Let $M_n(I)$ be a weak n-cyclic refined neutrosophic module over the ring R, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a weak n-cyclic refined neutrosophic submodule if and only if:

$x + y \in W_n(I), m.x \in W_n(I)$ for all $x, y \in W_n(I), m \in R$.

Proof:

It holds directly from the fact that is$W_n(I)$ is a submodule of $M_n(I)$.

Theorem 5.8:

Let $M_n(I)$ be a strong n-cyclic refined neutrosophic module over the n-cyclic refined neutrosophic ring $R_n(I)$, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a strong n-cyclic refined neutrosophic submodule if and only if:

$x + y \in W_n(I), m.x \in W_n(I)$ for all $x, y \in W_n(I), m \in R_n(I)$.

Proof:

It holds directly from the fact that is$W_n(I)$ is a submodule of $M_n(I)$ over the n-cyclic refined neutrosophic ring $R_n(I)$.

Example 5.9:

$M = R^2$ is a module over the ring of real numbers R, $W =< (0,1) >$ is a submodule of M, $R_2^2(I) = \{(a, b) + (m, s)I_1 + (k, t)I_2; a, b, m, s, k, t \in R\}$ is the corresponding weak/strong 2-cyclic refined neutrosophic module.

$W_2(I) = \{a_0 + a_1 I_1 + a_2 I_2\} = \{(0, x) + (0, y)I_1 + (0, z)I_2; x, y, z \in R\}$ is a weak 2-cyclic refined neutrosophic submodule of the weak 2-cyclic refined neutrosophic module $R_2^2(I)$ over the ring R.

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$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ is a strong 2-cyclic refined neutrosophic submodule of the strong 2-cyclic refined neutrosophic module $R_2^2(I)$ over the n-cyclic refined neutrosophic ring $R_2(I)$.

Definition 5.10:

Let $M_n(I)$ be a weak n-cyclic refined neutrosophic module over the ring R, x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \leq M_n(I)$ if$x = a_1x_1 + a_2x_2 + \cdots + a_mx_m$: $a_i \in R, x_i \in M_n(I)$.

Example 5.11:

Consider the weak 2-cyclic refined neutrosophic module in Example 3.9,

$x = (0,2) + (1,3)I_1 \in R_2^2(I)$, $x = 2(0,1) + 1.(1,0)I_1 + 3(0,1)I_1$, i.e x is a linear combination of the set $\{(0,1), (1,0)I_1, (0,1)I_1\}$ over the ring R.

Definition 5.12:

Let $M_n(I)$ be a strong n-cyclic refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \leq M_n(I)$ if$x = a_1x_1 + a_2x_2 + \cdots + a_mx_m$: $a_i \in R_n(I), x_i \in M_n(I)$.

Example 5.13:

Consider the strong 2-cyclic refined neutrosophic module $R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ over the 2-cyclic refined neutrosophic ring $R_2(I)$,

$x = (0,2) + (2,3)I_1 = (2 + I_1).(0,1) + (1 + I_2).(1,1)I_1 = (0,2) + (0,1)I_1 + (1,1)I_1 + (1,1)I_{(3 \bmod 2)} = x$, hence x is a linear combination of the set $\{(0,1), (1,1)I_1\}$ over the 2-cyclic refined neutrosophic ring $R_2(I)$.

Definition 5.14:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a weak n-cyclic refined neutrosophic module M_n(I) over the ring R, X is a weak linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0 \;\; implies \;\; a_i = 0; \; a_i \in R$.

Definition 5.15:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a strong n-cyclic refined neutrosophic module M_n(I) over the n-cyclic refined neutrosophic ring $R_n(I)$, X is a strong linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0 \; implies a_i = 0; \; a_i \in R_n(I)$.

Definition 5.16:

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Let $M_n(I), W_n(I)$ be two strong n-cyclic refined neutrosophic modules over the n-cyclic refined neutrosophic ring $R_n(I)$, let $f: M_n(I) \to U_n(I)$ be a well defined map. It is called a strong n-cyclic refined neutrosophic homomorphism if:

$f(a.x + b.y) = a.f(x) + b.f(y)$ for all $x, y \in M_n(I), a, b \in R_n(I)$.

A weak n-cyclic refined neutrosophic homomorphism can be defined by the same.

Definition 5.17:

Let $f: M_n(I) \to U_n(I)$ be a weak/strong n-cyclic refined neutrosophic homomorphism, we define:

(a) $Ker(f) = \{x \in M_n(I); f(x) = 0\}$.

(b) $Im(f) = \{y \in U_n(I);\ \exists\ x \in M_n(I)\ andy = f(x)\}$.

Theorem 5.18:

Let $f: M_n(I) \to U_n(I)$ be a weak n-cyclic refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a weak n-cyclic refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a weak n-cyclic refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a weak n-cyclic refined neutrosophic submodule of$M_n(I)$.

(b) Holds by similar argument.

Theorem 5.19:

Let $f: M_n(I) \to U_n(I)$ be a strong n-cyclic refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-cyclic refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a strong n-cyclic refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-cyclic refined neutrosophic ring $R_n(I)$, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a strong n-cyclic refined neutrosophic submodule of $M_n(I)$.

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(b) Holds by similar argument.

Theorem 5.20:

Let $f: M_n(I) \to U_n(I)$ be a strong n-cyclic refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-cyclic refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a strong n-cyclic refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-cyclic refined neutrosophic ring $R_n(I)$, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a strong n-cyclic refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Example 5.21:

Let $R_2^2(I) = \{x_0 + x_1I_1 + x_2I_2;\ x_0, x_1, x_2 \in R^2\}$, $R_2^3(I) = \{y_0 + y_1I_1 + y_2I_2;\ y_0, y_1, y_2 \in R^3\}$ be two weak 2-cyclic refined neutrosophic modules over the ring of real numbers R. Consider $f: R_2^2(I) \to R_2^3(I)$, where

$f[(a, b) + (m, n)I_1 + (k, s)I_2] = (a, 0,0) + (m. 0.0)I_1 + (k, 0,0)I_2$, f is a weak 2-cyclic refined neutrosophic homomorphism over the ring R.

$Ker(f) = \{(0, b) + (0, n)I_1 + (0, s)I_2; b, n, s \in R\}$.

$Im(f) = \{(a, 0,0) + (m, 0,0)I_1 + (k, 0,0)I_2; a, m, k \in R\}$.

Example 5.22:

Let $W_2(I) =< (0,0,1)I_1 >= \{q. (0,0, a)I_1; a \in R, q \in R_2(I)\}$, $U_2(I) =< (0,1,0)I_1 >= \{q. (0, a, 0)I_1; a \in R; q \in R_2(I)\}$ be two strong 2-cyclic refined neutrosophic modules of the strong 2-cyclic refined neutrosophic module $R_2^3(I)$ over 2-refined neutrosophic ring R_2(I). Define $f: W_2(I) \to U_2(I); f[q(0,0, a)I_1] = q(0, a, 0)I_1;\ q \in R_2(I)$.

f is a strong 2-cyclic refined neutrosophic homomorphism:

Let $A = q_1(0,0, a)I_1, B = q_2(0,0, b)I_1 \in W_2(I);\ q_1, q_2 \in R_2(I)$, we have

$$A + B = (q_1 + q_2)(0,0, a + b)I_1, f(A + B) = (q_1 + q_2). (0, a + b, 0)I_1 = f(A) + f(B).$$

Let $m = c + dI_1 + eI_2 \in R_2(I)$ be a 2-cyclic refined neutrosophic scalar, we have

$f(m. A) = m. f(A)$ by the definition of f, hence f is a strong 2-cyclic refined neutrosophic homomorphism.

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5. Conclusion

In this paper, we have defined the n-cyclic refined neutrosophic ring and n-cyclic refined neutrosophic polynomial ring as new generalizations of the concept of refined neutrosophic set, we have introduced and studied n-cyclic AH-structures such as:

n-cyclic AH-ideal, n-cyclic AHS-ideal, n-cyclic AH-weak principal ideal, n-cyclic AH-weak prime ideal.

Also, we have introduced the concept of n-cyclic refined neutrosophic modules as a direct aapplication of n-cyclic refined neutrosophic rings. In the future, we aim to define and study some n-cyclic refined neutrosophic algebraic structures such as n-cyclic refined neutrosophic vector spaces, and groups. Another application of n-cyclic refined system can be in the theory of decision making and neutrosophic programming in a similar way of classical refined neutrosophic set.

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Completeness and Compactness in Standard Single Valued Neutrosophic Metric Spaces

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Abstract

In a recent paper, we have introduced the notion of standard single valued neutrosophic metric space as a generalization of standard fuzzy metric spaces given by J.R. Kider and Z.A. Hussain. In this paper, we continue our previous work by introducing the notions of complete standard single valued neutrosophic metric space and compact standard single valued neutrosophic metric space. Furthermore, we give a number of properties and characterizations of these notions and relationship between them.
Keywords: Single valued neutrosophic set, Metric space, Completeness, Compactness

1 Introduction

In 1995, Smarandache proposed the notion of neutrosophic set, which was published in 1998 [10] as a generalization of the notions of fuzzy set and intuitionistic fuzzy set. A neutrosophic set (NS) is a set where each element of the universe has a degree of truth (T), indeterminacy (I) and falsity (F) in the non standard unit interval. Further, Wang et al. [15] proposed the notion of single valued neutrosophic set (SVNS) as a subclass of (NS). Single valued neutrosophic sets have been useful in many real applications in several branches (see for e.g., [2,5,6,7,13] and [17]).
In the literature, there are several approaches to the notion of neutrosophic metric space. In [14], Taş et al. defined the neutrosophic valued metric spaces and neutrosophic valued g-metric spaces. In this regard, we find that other authors have adopted the same approach, such as Şahin et al. [8,9]. Later on, Kirişci and Şimşek [4] introduced neutrosophic metric space with neutrosophic numbers and they investigated some properties of neutrosophic metric space such as compactness and completeness. In the present study, we introduce the notion of standard single valued neutrosophic metric space and, from this notion, we introduce the notion of complete standard single valued neutrosophic metric space and compact standard single valued neutrosophic metric space. Furthermore, we give a number of properties and characterizations of these notions and relationship between them.
This paper is structured as follows. In Section 2, we recall basic concepts and properties of single valued neutrosophic sets. Moreover, we introduce the concept of standard single valued neutrosophic metric spaces and some related notions that will be needed throughout this paper. In Section 3, we introduce the notion of complete single valued neutrosophic metric space and we show its interesting properties. In Section 4, we introduce the notion of compact single valued neutrosophic metric space with interesting characterizations and properties and relationship between completeness and compactness. Finally, we present some conclusions and we discuss future research in Section 5.

2 Preliminaries

This section contains the basic definitions and properties of single valued neutrosophic sets and some related notions that will be needed throughout this paper.

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2.1 Single valued neutrosophic sets

The notion of fuzzy sets was first introduced by Zadeh [18].

Definition 2.1. [18] Let X be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for any $x \in X$.

In 1983, Atanassov [1] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 2.2. [1] Let X be a nonempty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$ which satisfy the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \text{ for any } x \in X.$$

In 1998, Smarandache [10] defined the concept of a neutrosophic set as a generalization of Atanassov's intuitionistic fuzzy set. Also, he introduced neutrosophic logic, neutrosophic set and its applications in [11,12]. In particular, Wang et al. [15] introduced the notion of a single valued neutrosophic set.

Definition 2.3. [11] Let X be a nonempty set. A neutrosophic set (NS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to]^{-}0, 1^{+}[$ and an indeterminacy function $\sigma_A : X \to]^{-}0, 1^{+}[$ and a non-membership function $\nu_A : X \to]^{-}0, 1^{+}[$ which satisfy the condition:

$$^{-}0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^{+}, \text{ for any } x \in X.$$

Certainly, intuitionistic fuzzy sets are neutrosophic sets by setting $\sigma_A(x) = 1 - \mu_A(x) - \nu_A(x)$.
Next, we show the notion of single valued neutrosophic set as an instance of neutrosophic set which can be used in real scientific and engineering applications.

Definition 2.4. [15] Let X be a nonempty set. A single valued neutrosophic set (SVNS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a truth-membership function $\mu_A : X \to [0,1]$, an indeterminacy-membership function $\sigma_A : X \to [0,1]$ and a falsity-membership function $\nu_A : X \to [0,1]$.

The class of single valued neutrosophic sets on X is denoted by $SVN(X)$.

For any two SVNSs A and B on a set X, several operations are defined (see, e.g., [15,16]). Here we will present only those which are related to the present paper.

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$,

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\sigma_A(x) = \sigma_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$,

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$,

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$,

(v) $\overline{A} = \{\langle x, 1 - \nu_A(x), 1 - \sigma_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$.

2.2 Standard single valued neutrosophic metric spaces

In this Subsection, we extend the notion of standard fuzzy metric space introduced by J.R. Kider and Z.A. Hussain [3] to the setting of single valued neutrosophic sets. Also, we discuss the main properties related to this notion.

Definition 2.5. A quintuple $(X, M, *, \triangleleft, \diamond)$ is said to be a standard single valued neutrosophic metric space (SSVN-metric space, for short) if X is an arbitrary set, $*$, $\triangleleft$ are a continuous t-norms, $\diamond$ is a t-conorm and M is a continuous single valued neutrosophic set on X^2 satisfying the following conditions:

(i) $\mu_M(x,y) > 0$, $\sigma_M(x,y) > 0$ and $\nu_M(x,y) < 1$ for all $x, y \in X$;

(ii) $\mu_M(x,y) = 1$, $\sigma_M(x,y) = 1$ and $\nu_M(x,y) = 0$ if and only if $x = y$;

(iii) $\mu_M(x,y) = \mu_M(y,x),\ \sigma_M(x,y) = \sigma_M(y,x)$ and $\nu_M(x,y) = \nu_M(y,x)$ for all $x, y \in X$;

(iv) $\mu_M(x,z) \geq \mu_M(x,y) * \mu_M(y,z),\ \sigma_M(x,z) \geq \sigma_M(x,y) \triangleleft \sigma_M(y,z)$ and $\nu_M(x,z) \leq \nu_M(x,y) \diamond \nu_M(y,z)$.

The functions $\mu_M(x,y),\ \sigma_M(x,y)$ and $\nu_M(x,y)$ denote the degree of nearness, the degree of neutralness and the degree of non-nearness between x and y, respectively.

Remark 2.6. If the set X given in the previous definition is a metric space with an ordinary distance d, then $(X, M, *, \triangleleft, \diamond)$ is called an SSVN-metric space induced by (X, d).

Example 2.7. Let (X, d) be an ordinary metric space. Define the t-norms $x * y = min\{x, y\},\ x \triangleleft y = min\{x, y\}$ and the t-conorm $x \diamond y = max\{x, y\}$, for all $x,\ y \in [0,1]$. Define the single valued neutrosophic set M on X^2 as:
$\mu_M(x,y) = \frac{1}{1+d(x,y)},\ \sigma_M(x,y) = d(x,y),\ \nu_M(x,y) = \frac{d(x,y)}{1+d(x,y)}$.
Then, $(X, M, *, \triangleleft, \diamond)$ is an SSVN-metric space.

Definition 2.8. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. For $x \in X$ and $r \in]0,1[$, the open ball $\mathcal{B}(x,r)$ with radius r and center x is defined by

$$\mathcal{B}(x,r) = \{y \in X \mid \mu_M(x,y) > 1-r,\ \sigma_M(x,y) > 1-r \text{ and } \nu_M(x,y) < r\}.$$

Definition 2.9. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space, a subset A of X is said to be an open set (OS, for short) if for any $x \in A$ there exists $r \in]0,1[$ such that $\mathcal{B}(x,r) \subseteq A$. The complement of an open set is called a closed set (CS, for short) in X.

Definition 2.10. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. Then

(i) a sequence (x_n) in X is said to be convergent to a point $x \in X$ if for any $r \in]0,1[$, there exists $k \in \mathbb{N}$ such that

$$\mu_M(x_n, x) > 1-r, \sigma_M(x_n, x) > 1-r \text{ and } \nu_M(x_n, x) < r, \text{ for all } n \geq k\,.$$

(ii) a sequence (x_n) in X is said to be Cauchy sequence if for any $r \in]0,1[$, there exists $k \in \mathbb{N}$ such that

$$\mu_M(x_n, x_m) > 1-r, \sigma_M(x_n, x_m) > 1-r \text{ and } \nu_M(x_n, x_m) < r, \text{ for all } n, m \geq k\,.$$

Definition 2.11. Let $(X, M, *, \triangleleft, \diamond)$ ba an SSVN-metric space and let $A \subseteq X$ then the closure of A is denoted by $\overline{A}$ is defined by the set of all limit of sequences (x_n) in A.

Definition 2.12. Let $(X, M, *, \triangleleft, \diamond)$ ba an SSVN-metric space and let $A \subseteq X$. A is said to be dense in X if $\overline{A} = X$.

Remark 2.13. Let $r_1, r_2 \in [0,1]$. If $r_1 > r_2$, then there exist $r_3, r_4 \in]0,1[$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_2 \diamond r_4$. Moreover, for any $r_5 \in]0,1[$, there exist $r_6, r_7 \in]0,1[$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

3 Completeness in SSVN-metric spaces

In this section, we will study some interesting properties of completeness in single valued neutrosophic metric spaces. First, we introduce the notion of complete single valued neutrosophic metric space.

Definition 3.1. An SSVN-metric space in which every Cauchy sequence is convergent, is said to be complete.

Theorem 3.2. *If every Cauchy sequence in an SSVN-metric space $(X, M, *, \triangleleft, \diamond)$ has a convergent subsequences. Then $(X, M, *, \triangleleft, \diamond)$ is complete.*

Proof. Let (x_n) be a Cauchy sequence, and let (x_{i_n}) be a subsequence of (x_n) where x_{i_n} converges to x. Take $r \in]0,1[$ such that

$$(1-r) * (1-r) > 1-\alpha, (1-r) \triangleleft (1-r) > 1-\alpha \text{ and } r \diamond r < \alpha, \text{ for all } \alpha \in]0,1[\,.$$

Since x_{i_n} converges to x, there exists $i_p \in \mathbb{N}$ such that

$$\mu_M(x_{i_n}, x) > 1-r,\ \sigma_M(x_{i_n}, x) > 1-r \text{ and } \nu_M(x_{i_n}, x) < r, \text{ for all } i_n \geq i_p\,.$$

In fact that (x_n) is a Cauchy sequence, then there exists $k \in \mathbb{N}$ where $k \geq i_p$ such that

$$\mu_M(x_n, x_m) > 1 - r,\ \sigma_M(x_n, x_m) > 1 - r \text{ and } \nu_M(x_n, x_m) < r,\ \text{ for all } n,\ m \geq k\,.$$

Therefore, if $n \geq i_p$, then

$$\mu_M(x_n, x) \geq \mu_M(x_n, x_m) * \mu_M(x_m, x) > (1-r) * (1-r) > 1 - \alpha,$$

$$\sigma_M(x_n, x) \geq \sigma_M(x_n, x_m) \triangleleft \sigma_M(x_m, x) > (1-r) \triangleleft (1-r) > 1 - \alpha$$

$$\text{and } \nu_M(x_n, x) \leq \nu_M(x_n, x_m) \diamond \nu_M(x_m, x) < r \diamond r < \alpha\,.$$

Thus, we have x_n converges to x, and hence $(X, M, *, \triangleleft, \diamond)$ is complete. □

Next, we discuss the relationship between dense subset and completeness in SSVN-metric space. First, we need to provide the following key result.

Lemma 3.3. *Let A be an SSVN-metric space $(X, M, *, \triangleleft, \diamond)$. If A is dense in X, then there exists $a \in A$ such that*

$$\mu_M(x, a) > 1 - r,\ \sigma_M(x, a) > 1 - r \text{ and } \nu_M(x, a) < r,\ \text{ where } r \in]0, 1[\text{ and } x \in X\,.$$

Proof. Suppose that A is dense in X and let $x \in X$. Then, $x \in \overline{A}$, and hence there exists a sequence (a_n) in A such that a_n converges to x. Hence, for any $r \in]0, 1[$, there exists $k \in \mathbb{N}$ such that

$$\mu_M(a_n, x) > 1 - r,\ \sigma_M(a_n, x) > 1 - r \text{ and } \nu_M(a_n, x) < r,\ \text{ for all } n \geq k\,.$$

Now, if we take $a = a_k$, then

$$\mu_M(a, x) > 1 - r,\ \sigma_M(a, x) > 1 - r \text{ and } \nu_M(a, x) < r,\ \text{ for all } k \geq N\,.$$

This is the desired result. □

Theorem 3.4. *Let A be a dense subset of an SSVN-metric space $(X, M, *, \triangleleft, \diamond)$. If every Cauchy sequence of points of A converges in X then $(X, M, *, \triangleleft, \diamond)$ is complete.*

Proof. Let (x_n) be a Cauchy sequence in X. On the one hand, since A is dense, it follows from Lemma 3.3 that for every $x_n \in X$ there exists $a_n \in A$ such that

$$\mu_M(x_n, a_n) > 1 - s,\ \sigma_M(x_n, a_n) > 1 - s \text{ and } \nu_M(x_n, a_n) < s \text{ where } s \in]0, 1[\,.$$

On the other hand, from Remark 2.13 there exists $t = 1 - \varepsilon \in]0, 1[$, such that

$$(1-s) * (1-s) > t,\ (1-s) \triangleleft (1-s) > t \text{ and } s \diamond s < \varepsilon\,.$$

We next show that the sequence (a_n) is Cauchy.
Indeed, since (x_n) is Cauchy in X, it then follows that for any $r \in]0, 1[$, there exists $k \in \mathbb{N}$ such that

$$\mu_M(x_n, x_m) > t,\ \sigma_M(x_n, x_m) > t \text{ and } \nu_M(x_n, x_m) < \varepsilon \text{ for all } n, m \geq k\,.$$

Therefore,

$$\mu_M(a_n, a_m) \geq \mu_M(a_n, x_n) * \mu_M(x_n, a_m) > (1-s) * (1-s) > t,$$

$$\sigma_M(a_n, a_m) \geq \sigma_M(a_n, x_n) \triangleleft \sigma_M(x_n, a_m) > (1-s) \triangleleft (1-s) > t$$

$$\text{and } \nu_M(a_n, a_m) \leq \nu_M(a_n, x_n) \diamond \nu_M(x_n, a_m) < s \diamond s < \varepsilon\,.$$

Then (a_n) is Cauchy sequence and since A is dense of $(X, M, *, \triangleleft, \diamond)$ this implies that (a_n) is converges to $x \in X$. On the other hand, $\mu_M(x_n, x) \geq \mu_M(x_n, a_n) * \mu_M(a_n, x) \geq (1-s) * (1-s) \geq 1-\varepsilon$, $\sigma_M(x_n, x) \geq \sigma_M(x_n, a_n) \triangleleft \sigma_M(a_n, x) \geq (1-s) \triangleleft (1-s) \geq 1-\varepsilon$ and $\nu_M(x_n, x) \leq \nu_M(x_n, a_n) \diamond \nu_M(a_n, x) \leq (s \diamond s) \leq \varepsilon$
Then (x_n) is converges to x. Hence, $(X, M, *, \triangleleft, \diamond)$ is complete. □

Now, we introduce the notion of continuous mapping and uniformly continuous mapping in SSVN-metric spaces.

Definition 3.5. Let $(X, M_X, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic continuous at $a \in X$, if for every $r \in]0, 1[$, there exists $\delta \in]0, 1[$ such that

$$\mu_{M'}(f(x), f(a)) > 1 - r,\ \sigma_{M'}(f(x), f(a)) > 1 - r \text{ and } \nu_{M'}(f(x), f(a)) < r\,,$$

$$\text{whenever } \mu_M(x, a) > 1 - \delta,\ \sigma_M(x, a) > 1 - \delta \text{ and } \nu_M(x, a) < \delta\,.$$

Definition 3.6. Let $(X, M_X, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic uniformly continuous on X, if for every $r \in]0, 1[$, there exists $\delta \in]0, 1[$ such that

$$\mu_{M'}(f(x_1), f(x_2)) > 1 - r,\ \sigma_{M'}(f(x_1), f(x_2)) > 1 - r \text{ and } \nu_{M'}(f(x_1), f(x_2)) < r\,,$$

$$\text{whenever } \mu_M(x_1, x_2) > 1 - \delta,\ \sigma_M(x_1, x_2) > 1 - \delta \text{ and } \nu_M(x_1, x_2) < \delta\,.$$

Theorem 3.7. *Let $f : (X, M, *, \triangleleft, \diamond) \longrightarrow (Y, M', *, \triangleleft, \diamond)$ to be a one-to-one and uniformly continuous. If f^{-1} is a single valued neutrosophic continuous and Y is complete, then X is complete.*

Proof. Suppose (x_n) is a Cauchy sequence and let the sequence $y_n = f(x_n)$. We show that (y_n) is a Cauchy sequence. Since (x_n) is a Cauchy sequence, it follows that

$$\mu_M(x_1, x_2) > 1 - \delta,\ \sigma_M(x_1, x_2) > 1 - \delta \text{ and } \nu_M(x_1, x_2) < \delta\,,$$

for any $\delta \in]0, 1[$. This implies that

$$\mu_{M'}(f(x_1), f(x_2)) > 1 - r,\ \sigma_{M'}(f(x_1), f(x_2)) > 1 - r \text{ and } \nu_{M'}(f(x_1), f(x_2)) < r\,,$$

for any $r \in]0, 1[$ and, there exists $k \in \mathbb{N}$ such that $m, n > k$ imply that

$$\mu_M(x_n, x_m) > 1 - \delta,\ \sigma_M(x_n, x_m) > 1 - \delta \text{ and } \nu_M(x_n, x_m) < \delta\,.$$

It follows that for $m, n > k$

$$\mu_{M'}(y_n, y_m) > 1 - r,\ \sigma_{M'}(y_n, y_m) > 1 - r \text{ and } \nu_{M'}(y_n, y_m) < r\,.$$

Hence, (y_n) is Cauchy sequence which implies that there exists a subsequence (y_{n_k}) such that y_{n_k} converge to y, where $y \in Y$. Since f^{-1} is a single valued neutrosophic continuous mapping, it follows that $x_{n_k} = f^{-1}(y_{n_k})$ converges to $f^{-1}(y) = x$. According to Theorem 3.2, X is complete. □

4 Compactness in SSVN-metric spaces

In this section, we will study some interesting properties and characterizations of compactness in single valued neutrosophic metric spaces.

4.1 Definitions

In this subsection, we introduce the notion of SVN-bounded subset, totally bounded subset and compact single valued neutrosophic metric space.

Definition 4.1. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$. A is said to be an SVN-bounded if there exists $r \in]0, 1[$ such that $\mu_M(x, y) > 1 - r,\ \sigma_M(x, y) > 1 - r$ and $\nu_M(x, y) < r$, for all $x, y \in A$.

Definition 4.2. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$.
A collection $\mathcal{O}$ of open sets is called an open cover of A if, $A \subseteq \underset{U \in \mathcal{O}}{\cup} U$.

Definition 4.3. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$. A is said to be a totally bounded if there exists $r \in]0, 1[$ such that $\mu_M(x, y_i) > 1 - r,\ \sigma_M(x, y_i) > 1 - r$ and $\nu_M(x, y_i) < r$, for all $x \in X$ and $y_i \in A$ with $i = 1, ..., n$.

Definition 4.4. Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space.
X is said to be compact if, $X = \overset{n}{\underset{i=1}{\cup}} U_i \mid U_i \subseteq \mathcal{O}$. In other words, if every open cover has a finite subcover.

4.2 Characterizations of compact SSVN-metric spaces

In this subsection, we provide interesting characterizations of compact SSVN-metric spaces.

Proposition 4.5. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$. If A is a totally bounded, then A is an SVN-bounded.*

Proof. Assume that A is a totally bounded subset of X, and consider an open cover $\{\mathcal{B}(x, r), x \in A\}$ of A. Since A is a totally bounded, there exist $x_1, x_2, ..., x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} \mathcal{B}(x_i, r)$. Let $x, y \in A$, then $x \in \mathcal{B}(x_i, r)$ and $y \in \mathcal{B}(x_j, r)$, for some $1 \leq i, j \leq n$. Consequently,

$$\mu_M(x_i, x) > 1 - r,\ \sigma_M(x_i, x) > 1 - r \text{ and } \nu_M(x_i, x) < r$$

$$\text{and } \mu_M(x_j, y) > 1 - r,\ \sigma_M(x_j, y) > 1 - r \text{ and } \nu_M(x_j, y) < r.$$

Due to the symmetry of the functions μ_M, σ_M and ν_M (see (iii) of Definition 2.5), it holds that

$$\mu_M(x, x_i) > 1 - r,\ \sigma_M(x, x_i) > 1 - r \text{ and } \nu_M(x, x_i) < r$$

$$\text{and } \mu_M(y, x_j) > 1 - r,\ \sigma_M(y, x_j) > 1 - r \text{ and } \nu_M(y, x_j) < r.$$

Now, we put $r_0 = \min\{\mu_M(x, x_i); 1 \leq i, j \leq n\}$, $r_1 = \min\{\sigma_M(x, x_i); 1 \leq i, j \leq n\}$ and $r_2 = \max\{\nu_M(x, x_i); 1 \leq i, j \leq n\}$. Then there exists $s \in]0, 1[$ such that $r_0 > 1 - s > 1 - r$, $r_1 > 1 - s > 1 - r$ and $r_2 < s < r$. Moreover, we obtain

$$\mu_M(x, y) \geq \mu_M(x, x_i) * \mu_M(x_i, x_j) * \mu_M(x_j, x) \geq (1 - r) * (1 - r) * r_0 > 1 - s > 1 - r,$$

$$\sigma_M(x, y) \geq \sigma_M(x, x_i) \triangleleft \sigma_M(x_i, x_j) \triangleleft \sigma_M(x_j, x) \geq (1 - r) \triangleleft (1 - r) \triangleleft r_1 > 1 - s > 1 - r$$

$$\text{and } \nu_M(x, y) \leq \nu_M(x, x_i) \diamond \nu_M(x_i, x_j) \diamond \nu_M(x_j, x) \leq r \diamond r \diamond r_2 < s < r.$$

Therefore,

$$\mu_M(x, y) > 1 - r,\ \sigma_M(x, y) > 1 - r \text{ and } \nu_M(x, y) < r \text{ for all } x, y \in A.$$

Hence, A is an SVN-bounded. □

Proposition 4.6. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. If X is compact, then X is totally bounded.*

Proof. It is clear that for any given $r \in]0, 1[$, the collection $\mathcal{O}$ of all balls $\mathcal{B}(x, r)$ is an open cover of X, where $x \in X$. Let X be a compact SSVN-metric space. Since X is compact, it follows that $\mathcal{O}$ contains a finite subcover. Hence, for $r \in]0, 1[$, there exists a finite number of open balls $\mathcal{B}(x_i, r)$ which represents an open cover of X, where $i = 1, 2, ..., n$. Now, if we consider $x \in \mathcal{B}(x_i, r)$ then

$$\mu_M(x, x_i) > 1 - r,\ \sigma_M(x, x_i) > 1 - r \text{ and } \nu_M(x, x_i) < r, \text{ for } i = 1, 2, ..., n.$$

Therefore, X is totally bounded. □

Combining Proposition 4.6 and Proposition 4.5 easily leads to the following result.

Corollary 4.7. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space and let $A \subseteq X$. If A is compact, then A is an SVN-bounded.*

Proposition 4.8. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. If X is compact, then X is complete.*

Proof. Suppose that X is compact and consider at the same time that X is not complete. Since X is not complete, there exists a Cauchy sequence (x_n) not having a limit in X. Now, we assume that $x \in X$. Since (x_n) does not converge to x, there exists $r_1 \in]0, 1[$ such that

$$\mu_M(x_n, x) \leq 1 - r_1,\ \sigma_M(x_n, x) \leq 1 - r_1 \text{ and } \nu_M(x_n, x) \geq r_1 \text{ for all } x, y \in A \text{ for any } n \in \mathbb{N}.$$

In addition, as long as (x_n) is Cauchy, there exists an integer $k \in \mathbb{N}$ such that $n, m \geq k$. This implies that

$$\mu_M(x_n, x_m) > 1 - r_2,\ \sigma_M(x_n, x_m) > 1 - r_2 \text{ and } \nu_M(x_n, x_m) < r_2, \text{ where } r_2 \in]0, 1[.$$

Next, we choose $m \geq k$ for which

$$\mu_M(x_m, x) > 1 - r_2,\ \sigma_M(x_m, x) > 1 - r_2 \text{ and } \nu_M(x_m, x) < r_2.$$

Then, the open ball $\mathcal{B}(x, r_2)$ contains x_n for only a finite number of values of $n \geq k$. Thus, it can be observed that $X = \overset{n}{\underset{x \in X}{\cup}} \mathcal{B}(x, r)$, which means that $\mathcal{O} = \{\mathcal{B}(x, r), x \in X\}$ is an open cover of X. The compactness of X implies that this open cover contains a finite subcover $\mathcal{O}_i = \{\mathcal{B}(x_i, r_i), x_i \in X, i = 1, 2, ..., n\}$.
In conclusion, as each ball contains (x_n) for only a finite number of values of $n \geq k$, the balls are in $\mathcal{O}_i$. In addition, X must contains (x_n) also for only a finite number of values of $n \geq k$ which means that a Cauchy sequence (x_n) have a limit in X. This, contradicts the hypothesis. Hence X is complete. □

Proposition 4.9. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. If X is totally bounded and complete, then X is compact.*

Proof. Suppose that X is totally bounded and complete and consider at the same time that X is not compact. On the one hand, since X is not compact, then there exists an open cover $U_i, i \in I$ of X which does not contain a finite subcover. On the other hand, since X is totally bounded, then it follows from Proposition 4.5 that $X \subseteq \mathcal{B}(x, r)$, for any $x \in X$ and $r \in]0, 1[$. While it is clear that $\mathcal{B}(x, r) \subseteq X$, this implies that $X = \mathcal{B}(x, r)$.
Now, Setting $\alpha_n = \frac{r}{2^n}$. According to what we know that X being totally bounded can be covered by finite many balls of radius α_1, then from our hypothesis at least one of these balls, and so be it $\mathcal{B}(x_1, \alpha_1)$, cannot be covered by a finite number of sets U_i. As $\mathcal{B}(x_1, \alpha_1)$ is totally bounded, then we can find an $x_2 \in \mathcal{B}(x_1, \alpha_1)$ such that $\mathcal{B}(x_2, \alpha_2)$ cannot be covered by a finite number of sets U_i. Proceeding in this way, a sequence (x_n) can be defined with the property that for each n, $\mathcal{B}(x_n, \alpha_n)$ cannot be covered by a finite number of sets U_i and $x_{n+1} \in \mathcal{B}(x_n, \alpha_n)$.
Next, we show that the sequence (x_n) is convergent. The fact that $x_{n+1} \in \mathcal{B}(x_n, \alpha_n)$ implies that

$$\mu_M(x_n, x_{n+1}) > 1 - \alpha_n,\ \sigma_M(x_n, x_{n+1}) > 1 - \alpha_n \text{ and } \nu_M(x_n, x_{n+1}) < \alpha_n.$$

Similarly, $x_m \in \mathcal{B}(x_{m-1}, \alpha_{m-1})$ implies that

$$\mu_M(x_{m-1}, x_m) > 1 - \alpha_{m-1},\ \sigma_M(x_{m-1}, x_m) > 1 - \alpha_{m-1} \text{ and } \nu_M(x_{m-1}, x_m) < \alpha_{m-1}.$$

Let $\alpha \in]0, 1[$ such that

$$(1 - \alpha_n) * (1 - \alpha_{n+1}) * ... * (1 - \alpha_{m-1}) > 1 - \alpha\,,$$
$$(1 - \alpha_n) \triangleleft (1 - \alpha_{n+1}) \triangleleft ... \triangleleft (1 - \alpha_{m-1}) > 1 - \alpha$$
$$\text{and } \alpha_n \diamond \alpha_{n+1} \diamond ... \diamond \alpha_{m-1} < \alpha\,.$$

Therefore,

$$\begin{aligned} \mu_M(x_n, x_m) &\geq \mu_M(x_n, x_{n+1}) * \mu_M(x_{n+1}, x_{n+2}) * ... * \mu_M(x_{m-1}, x_m) \\ &> (1 - \alpha_n) * (1 - \alpha_{n+1}) * ... * (1 - \alpha_{m-1}) \\ &> 1 - \alpha\,, \end{aligned}$$

Applying a similar reasoning, we find

$$\begin{aligned} \sigma_M(x_n, x_m) &\geq \sigma_M(x_n, x_{n+1}) \triangleleft \sigma_M(x_{n+1}, x_{n+2}) \triangleleft ... \triangleleft \sigma_M(x_{m-1}, x_m) \\ &> (1 - \alpha_n) \triangleleft (1 - \alpha_{n+1}) \triangleleft ... \triangleleft (1 - \alpha_{m-1}) \\ &> 1 - \alpha \text{ and} \end{aligned}$$

$$\begin{aligned} \nu_M(x_n, x_m) &\leq \nu_M(x_n, x_{n+1}) \diamond \nu_M(x_{n+1}, x_{n+2}) \diamond ... \diamond \nu_M(x_{m-1}, x_m) \\ &< \alpha_n \diamond \alpha_{n+1} \diamond ... \diamond \alpha_{m-1} \\ &< \alpha\,. \end{aligned}$$

□

Hence, (x_n) is a Cauchy sequence. Since X is complete, then (x_n) converges to y in X. As $y \in X$, there exists $i_0 \in I$ such that $y \in U_{i_0}$. As U_{i_0} is open, then it contains $\mathcal{B}(y, \beta)$ where $\beta \in]0, 1[$, and hence, for n so large we have

$$\mu_M(x_n, y) > 1 - \beta,\ \sigma_M(x_n, y) > 1 - \beta \text{ and } \nu_M(x_n, y) < \beta \text{ with } 1 - \alpha_n > 1 - \beta \text{ and } \alpha_n < \beta\,.$$

Let $x \in \mathcal{B}(x_n, \alpha_n)$. It holds that

$$\mu_M(x, x_n) > 1 - \alpha_n,\ \sigma_M(x, x_n) > 1 - \alpha_n \text{ and } \nu_M(x, x_n) < \alpha_n \text{ for any } x \in X\,.$$

Thus,

$$\mu_M(x, y) \geq \mu_M(x, x_n) * \mu_M(x_n, y) \geq (1 - \beta) * (1 - \beta) > 1 - r\,,$$

$$\sigma_M(x, y) \geq \sigma_M(x, x_n) \triangleleft \sigma_M(x_n, y) \geq (1 - \beta) \triangleleft (1 - \beta) > 1 - r \text{ and}$$

$$\nu_M(x, y) \leq \nu_M(x, x_n) \diamond \nu_M(x_n, y) < \beta \diamond \beta < r\,.$$

This implies that $x \in \mathcal{B}(y, r)$, and hence $\mathcal{B}(x_n, \alpha_n) \subseteq \mathcal{B}(y, r)$. This means that $\mathcal{B}(x_n, \alpha_n)$ admits U_{i_0} as a finite subcover. This is a contradiction. Hence X is compact.

Theorem 4.10. *Let $(X, M, *, \triangleleft, \diamond)$ be an SSVN-metric space. Then it holds that X is compact if and only if X is totally bounded and complete.*

Proof. Suppose that X is compact. From Proposition 4.6, it then follows that X is totally bounded. Moreover, Proposition 4.8 then guarantees that X is complete. Thus, X is totally bounded and complete. The converse implication, follows immediately from Proposition 4.9. □

5 Conclusion

In this paper, we have introduced the notions of complete standard single valued neutrosophic metric space and compact standard single valued neutrosophic metric space and we have investigated their most interesting properties and characterizations. In a future work, we plan to study other topological properties for standard single valued neutrosophic metric space such as convexity, connexity and density. Moreover, we intend to use these topological properties to study some fixed point theorems.

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Applications of Neutrosophic Logic of Smart Agriculture via Internet of Things

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Abstract

This study presents a smart agriculture mechanism model equipped with neutrosophy theory for the first time. The model is created by meticulously bringing together the fields of decision making, IoT and cloud computing. We have demonstrated that smart agriculture can be used in integration with neutrosophic, integrated with IoT. This integration is a model created by making much more detailed calculations by taking into account and using the uncertainty situations in neutrosophic numbers and logic, by automating both the geometric analysis of the soil and surface control and the numerical data of the environment for smart agriculture.

Keywords: Smart Agriculture, Internet of Things, Cloud Computing, Neutrosophic Logic, Neutrosophic Geometry

1 Introduction

The term IoT (Internet of Things) was first used by Ashton, director of the Auto-ID Center. He defined that "Objects of The Internet (IoT) is a communication network in which physical objects are connected to each other or to larger systems. This network, from every moment of life it is full of technological devices that collect billions of data, turning it into meaningful information" [1]. IoT technology, electronics it has gained a new dimension thanks to miniaturization and advances in network connections. With these developments, IoT wireless is a network of advanced sensors that can communicate with each other over the internet and can be accessed through web technologies over the internet network it has become a technology in which it is used. IoT data collection, transmission, processing, business management, etc. currently and next-generation information it is a significant part of their technology and has opened a new way for people to collect and process information [2]. Within the fundamental IoT, natural environment includes temperature, position, weight, light intensity, pulse rate, blood pressure, hardness, carbon dioxide ratio, humidity, ph value, sound intensity, etc. all kinds like there is an environment where there are measurable physical magnitudes. These data are received through sensors are detected raw and analog or they are converted to digital signals. These data from nature are the RFID needed for IEEE 802.5.4, Bacnet, Bluetooth, GPRS and GSM, infrared, LPWAN, M-Bus, machine-machine communication, Man-Machine, ModBus, NFC, power line carriers, Zigbee, ethernet and it is transmitted to cloud computing systems for storage, such as for processing with wireless and wired communication infrastructure and communication protocols. The data store form big data in increasing stacks. This big amount of data to increase efficiency needs to be analyzed, and this is done using machine learning methods or the established rule base. IoT with the growth and spread of technology, traditional agriculture will be abandoned and switched to modern and more efficient agriculture. Also for industries, it will provide great business opportunities. These developments provide important groundwork for the development and innovation of Agriculture 4.0.

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Internet of Things (IoT) is a technology that has been growing in popularity in recent years. IoT is defined as the way objects communicate with each other through advanced communication technologies. In the future, thanks to developed projects such as LTE (Long-Term Evolution), M2M (machine-to-machine communication), today's most devices in use will be able to access an IP. The importance of IoT technology will bring many innovations increasing day by day via many similar developments. The industry 4.0 solutions, can achieve information about the place of IoT technology in smart agriculture is provided [3]. However, there is no specific consensus on the definition of IoT by the community of users around the world among academics, researchers, people from many different segments, including practitioners, developers, and legal entities. What all the definitions have in common is that the first version of the internet was created by people the idea is that it's about the data created, and the next version is about the data created by generations. The best definition for IoT "A clear and comprehensive smart objects capable of reacting and behaving in the face of changes in situations and the environment the network is stated as" [4]. IoT technology generally consists of detection, transmission and application layers. This technology has made measurable information about all kinds of objects that you can think of, remotely detected and monitored. Detection sensors are used to collect data at the layer. Sensors for fixed or mobile use as needed it can be designed. Electrical conductivity (EC), CO2 ratio, soil and, air humidity in the agricultural area discussed in this study today, temperature, light sensors are some of them. In recent years, passive, battery-free, non-contact, electrical resistance measurement sensors and studies are also being carried out on sensors that detect the physiological state of plants [5, 6]. Wireless devices with sensor networks (KSA) and RFID technologies and traditional wireless communication protocols such as WiFi, Bluetooth, GPRS in addition, the IEEE 802.15.4 standard and next-generation protocols such as Zigbee are used designed for this purpose at the transmission layer. The software layer is web-based mobile applications, especially microprocessor (MCU) software, server and cloud computing management software with it covers inter-machine communication protocol (M2M) software.

Agricultural system optimization is one of the significant factors for the recent economy to improve since countries are fully dependent on the factor that the today's resource management in agriculture is the most complicated, such as the water resource; the agriculture uses approximately 65 per cent of the fresh water; hence, water management is the most critical part in the agriculture to accomplish; hence, to achieve this, there are certain areas in agriculture where it is dependent such as the crop section for the soil, and in this factor, a PH sensor is used to measure the acidity in the solution; an IoT gate way is used to connect the field devices with the wireless Internet networks which could be wide area network (WAN) or a remote controlled device; the device can be an integration of IoT with cloud computing and sensors, and other resource combined will be managed by the cloud computing along with raspberry pi sensor which is a module to which various sensors such as the temperature and humidity are interfaced and also other will be the provision for the GSM-based automatic irrigation system for the efficient use of resources and crop planning; and when it comes to the operation of the system initially, the user registration is compulsory for the verification, and then, the admin checks the data provided by the user and allows to control the system; raspberry pi kit takes the sensor values and sends to Google spread sheet attached to the raspberry pi kit, the Google spread sheets used for maintaining real-time information and a data related to crop saved on these sheets, and then with ANN algorithm, sensor values maintain on Google spread sheets, compare according to threshold values, and check which crop is suitable for soil; this process is distributed in modules, this proposed system is best suitable for the crop selection based on the soil quality, and unwanted water wastage things are completely eliminated [7].

Cloud computing, or in its functional sense, online information distribution is the general name given to services that provide common information sharing between computing devices. In this respect, cloud computing is not a product, but a service. It is the sharing of software and information from the underlying source, and the use of existing information services over the information network (typically from the Internet) from computers and other devices in a similar way to electricity distributors. Zhao proposed a system that allows farmers to remotely monitor the data of the environment and plants obtained from greenhouses by sending them to mobile devices via the internet via sensors [8]. Duan discussed on the characteristic agricultural data which agricultural information management system can be used for agricultural production processes [9]. In the study, information about the smart agricultural management information system was given to make sensitive management decisions on issues such as crop cultivation, fertilization and input costs. Ying and Hao used IOT technology with the help of cloud computing method and proposed a smart cloud computing method to process data from different IOT devices [10]. They reported that the obtained data can be stored in cloud data centers and these data can be processed using smart large servers to solve the necessary problems. Qirui proposed an agricultural information service model based on agricultural expert systems in order to provide accurate and efficient agricultural information services to users through web browsers [11]. Hori et al. gave information about the cloud system established by Fujitsu ltd for the agricultural sector and designed a cloud model for agricultural applications. In addition, they explained how the products should be priced and sold

for the farmers using the developed model. Researchers have used prototype web and mobile phone applications for production-sales planning, operational planning management, information support and cultivated land data management functions [12]. The researchers reported that the PDCA (Plan-Do-Check-Act) cycle used in agricultural applications will make significant contributions to the development of cloud services.

One of the factors that spoils the standard of agricultural production is that the soil structure varies from region to region. Apart from region to region variability, no land has a homogeneous structure in itself. When a production area is examined carefully, it can be easily observed that the plants develop very well in places, they are weak in places, even drying and deaths occur in places. So, what is it that creates these differences within the same land? Of course, the reason for these differences is the physical, chemical and biological structure of the soil that can change in every decare, even every square meter. In fact, this variability lies in the emergence of many elements of smart agriculture. Smart agriculture is a technique that enables soil and product management to increase agricultural productivity, and minimize the damage to the environment by using resources more economically. In this context, it is aimed to give up classical production and to implement an application method that handles the land with an in-homogeneous and variable approach. The main factor aimed here is to use the inputs applied in agricultural production where they are needed, when and in quantity. Smart agriculture is a modern agricultural production technology that is based on the intervention to be made to the needs that differ spatially and temporally in the area where the crop is cultivated in an agricultural enterprise, taking these location and time criteria into consideration. Intelligent agriculture aims to prevent waste of resources, increase the gross yield of the product and minimize the environmental pollution caused by production with the use of improved information and control systems. Intelligent agriculture techniques can be used in almost every period of crop production from tillage to harvest. Among the goals of smart agriculture is to reduce chemical costs such as fertilizer and medicine; protecting the environment by reducing these uses; providing high quantity and quality products; Ensuring a more efficient flow of information for business and breeding decisions and establishing a record order in agriculture. It is aimed to maximize productivity with the Internet of Things in agriculture. As natural resources are used at the required level, the cost is reduced. Similarly, with the smart systems on the farm, all factors required for production are analyzed and presented to the producer simultaneously. In this way, resource wastage is prevented and quality products are produced. In addition, rapid decision-making mechanisms are created with machines that communicate with each other and work synchronously. Producers are given the opportunity to manage and monitor the entire farm from a tablet or phone, and by reducing the work force, and also more productivity, quality and natural production opportunities are created.

Applications of IoT in agriculture science decrease the human time, scales down the human efforts, and provide the easiness and effectiveness for the mechanism. Suppose that we apply in the irrigation area, we put the humidity sensors and the threshold sensors send the information to your mobile phone and then connect to the IoT when the humidity range increases, then you can turn off the water meter. There are many categories of industries that use these technologies for improved performance, such as automobiles, communications, food, medical, marine, defense and the most important agricultural industries. The use of these technologies contributes to food safety, environmental sustainability, good agricultural practices (GAP), computer artificial intelligence, decision-making, GPS and increased agricultural efficiency, as well as to crop protection and integrated management of pests and where possible crop management of pests and diseases. The application equipment can be chosen depending on these variables and the application equipment meets microbial protection so that the agriculture equipment will be selected. In addition, new innovations are being launched day by day and their implementations have also steadily expanded. The area of agriculture has tremendous potential for emerging technology such as artificial intelligence (AI) and the Internet of Things (IoT) to be implemented. By integrating this, smart systems would be able to make choices based on prior experience and understanding. This operation is made simpler by smart and automated machines. The AI contains many problem-solving logics and approaches such as fuzzy logic, artificial neural networks (ANN), neuro-fuzzy logic, and expert systems, of which the ANN approach is more commonly used and prescribed. Recently, fuzzy logic has found many applications and efficient results in smart agriculture and IoT [13, 14, 15, 16, 17]. We will use neutrosophic logic [18] in this study, which is a useful and practical extension of fuzzy logic. The topic of indeterminacy beyond the principles of fuzzy logic is often taken into account by neutrosophic logic.

2 Neutrosophic Logic Fundamentals

In classical set theory, an element either belongs to a set or not. The membership of elements in a set is interpreted in binary terms according to a divalent case. In fuzzy set theory, introduced by Zadeh [19], a gradual assessment of the membership of elements in a set is permitted by a membership function which takes

values in the real unit interval $[0, 1]$. In fuzzy set theory, classical divalent sets are usually called crisp sets. Fuzzy set theory is a generalization of the classical set theory. IFS are sets whose elements have degrees of membership and non-membership. IFS have been introduced by Atanassov [20] as an extension of the notion of fuzzy set, which itself extends the classical notion of a set. Neutrosophic set theory is a generalization of intuitionistic fuzzy set, crisp set, fuzzy set, para-consistent set, dialetheist set, paradoxist set, tautological set based on Neutrosophy [18]. An element $x(T, I, F)$ belongs to the set in the following way: it is true in the set with a degree of $t \in [0, 1]$, indeterminate with a degree of $i \in [0, 1]$, and it is false with a degree of $f \in [0, 1]$.

We will now give some definitions of the fundamental concepts related to our study.

Definition 2.1. [19] Given a universal set U and a generic element, denoted by x, a *fuzzy set* X in U is a set of ordered pairs defined as

$X = \{(x, \mu_X(x))|x \in U\}$, where $\mu_X : U \longmapsto [0, 1]$ is called the *membership function* of A and $\mu_X(x)$ is the *degree of membership* of the element x in X.

Definition 2.2. [20] An *intuitionistic fuzzy set* X over a universe of discourse U is represented as

$X = \{(x, \mu_X(x), \nu_X(x))|x \in U\}$, where $\mu_X : U \longmapsto [0, 1]$ and $\nu_X : U \longmapsto [0, 1]$ are called respectively the *membership function* of A and the *non-membership function* of A for x in X. The *degree of non-membership* of the element x in X is defined as $\mu_X(x) = 1 - \nu_X(x)$.

Definition 2.3. [18, 21] Let U be a universe of discourse. A *neutrosophic set* is defined as

$$N = \{(x, T(x), I(x), F(x)) : x \in U\},$$

which is identified by a *truth-membership function* $T_N : U \longmapsto]0^-, 1^+[$, *indeterminacy-membership function* $I_N : U \longmapsto]0^-, 1^+[$ and *falsity-membership function* $F_N : U \longmapsto]0^-, 1^+[$.

Definition 2.4. [22, 23] Let U be a universe of discourse. A *single valued neutrosophic set* is defined as

$$N = \{(x, T(x), I(x), F(x)) : x \in U\},$$

which is identified by *a truth-membership function* $T_N : U \longmapsto [0, 1]$, *indeterminacy-membership function* $I_N : U \longmapsto [0, 1]$ and *falsity-membership function* $F_N : U \longmapsto [0, 1]$ with $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3$. A *single-valued neutrosophic number* (SVNN) is denoted by $a = (T, I, F)$.

Definition 2.5. [18, 21] Let $A^* = \{(x, T(x), I(x), F(x)) : x \in A\}$ and $B^* = \{(y, T(y), I(y), F(y)) : y \in B\}$ be neutrosophic elements. $NR = \{((x, y), T(x, y), I(x, y), F(x, y)) : (x, y) \in A \times B\}$ is a neutrosophic relation on A^* and B^*.

2.1 Geometric Modeling on Neutrosophic Sets

Geometric modeling has a wide range of applications. Although mathematical models are expressed with differential equations, nowadays, geometric modeling tools are used as modeling tools [24]. One of them is Bèzier curves. In neutrosophic sets one can apply this to visualize data from a given neutrosophic set. Neutrosophic Bèzier curves are generated based on the control points from one of $TC = \{(x, y, T(x, y))\}, IC = \{(x, y, I(x, y))\}$ and $FC = \{(x, y, F(x, y))\}$ sets. Thus, there will be three different neutrosophic Bèzier curve models for a neutrosophic relation and variables x and y. A neutrosophic control point relation can be defined as a set of $\mathrm{n}+1$ points that shows a position and coordinate of a location and is used to described three curve which are denoted by $NR_{p_i} = \{NR_{p_0}, NR_{p_1}, \ldots, NR_{p_n}\}$ and can be written as

$$\{((x_0, y_0), T(x_0, y_0), I(x_0, y_0), F(x_0, y_0)) \ldots, \ldots, ((x_n, y_n), T(x_n, y_n), I(x_n, y_n), F(x_n, y_n))\}$$

in order to control the shape of a curve from a neutrosophic data.

Definition 2.6. A neutrosophic Bezier curve with degree n was defined by Taş and Topal [22] .

$$NB(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i NR_{p_i}, t \in [0, 1]$$

Definition 2.7. Neutrosophic Bézier surfaces [23] are generated by the control points from one of sets

$$TC = \{(x, y, T(x, y)) : x \in A, y \in B\}$$
$$IC = \{(x, y, I(x, y)) : x \in A, y \in B\}$$
$$FC = \{(x, y, F(x, y)) : x \in A, y \in B\}$$

Thus, there will be three different Bézier surface models for a neutrosophic relation and variables x and y. A neutrosophic control point relation can be defined as a set of $(n+1)(m+1)$ points that shows a position and coordinate of a location and is used to describe three surface which are denoted by

$$NR_{P_{ij}} = \{NR_{P_{00}}, NR_{P_{01}}, \ldots, NR_{P_{\text{num}}}\}$$

and can be written as quadruples

$$\{((x_i, y_j), T(x_i, y_j), I(x_i, y_j), F(x_i, y_j)) : i = \mathbf{0}, \ldots, n, j = \mathbf{0}, \ldots, m\}$$

in order to control the shape of a curve from a neutrosophic data.

Definition 2.8. A neutrosophic Bézier surface [23] with degree $n \times m$ is defined by

$$\mathrm{NB}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i} (1-u)^{n-i} u^i \binom{m}{j} (1-v)^{m-j} v^i \mathrm{NR}_{p_{ij}}$$

Every set of TC, IC and FC determines a Bézier surface. Thus, we obtain three Bézier surfaces. A neutrosophic Bézier surface is defined by these three surfaces. So it is a set of surfaces as in its definition.

3 An Application in IoT and Smart Agriculture

In agriculture, topography study has an important place for the operation of the photosynthesis mechanism depending on the geographical location of the field and the climate type. The mechanism of photosynthesis is possible with the maintenance of factors such as sunlight, CO_2, H_2O, heat, glucose, and O_2 in an optimal process. Along with these, the wavelength of light, humidity and soil properties play an important role in the production efficiency of the plant. Especially when focused on sensitive agriculture field, the details of the data and precise measurements require positioning and application for the analysis of big data and the interpretation of these data and values is a difficult process. In this sense, we give some applications that are neutrosophic useful and manageable tools in IoT and Smart Agriculture in this section.

3.1 Neutrosophic Inference Engine Design in Smart Agriculture and IoT

Here, we give neutrosophic methods for data gathering and evaluating in Neutrosophic Inference Engine Design (NIED).

Remark 3.1. One needs to have the following three properties for a possible NIED. After the properties constructed, The next step will be built a NIED.

1. Neutrosophication and De-neutrosophication

   While doing data analysis with fuzzy, fuzzification of crisp data and then defuzzification is required. Using neutrosophy for more precise analysis, measurement and evaluation of data should also provide what has been done for fuzzy. Recently, different types of neutrosophic numbers have been proposed for modeling different applications and fields of study. For example, linear pentagonal neutrosophic and its de-neutrosophication was introduced by [25]. The authors gave a spanning tree graph property and removal area method for its de-neutrosophication.With this graph analysis, pentagonal structures can be evaluated smoothly. Another example is trapezoidal neutrosophic number. Chakraborty et al. analysed various types of linear and non-linear generalized trapezoidal neutrosophic numbers and introduced its de-neutrosophication technique in the paper [26]. They used removal area and mean interval method for purpose of de-neutrosophication. However, in agriculture like study era, there are many factors and parameters to be taken account. So, it is easy to see that neutrosophic numbers having different dimensions will be demanded. Therefore, there is a serious need occurring for a general form of de-neutrosophication for n-gonal neutrosophic numbers. We state it as a challenging open problem.

2. Data Clustering and Analysis

   Data classification (henceforth, clustering) is one of the most vital steps of data assessment and manipulation. Rashno et al [27] used neutrosophic sets to handle boundary and outlier points as challenging points of clustering methods. Li et al [28] introduced a novel suitable objective function, which is depicted as a constrained minimization problem based on a single-valued neutrosophic set, is built, and the

Lagrange multiplier method is used to solve the objective function. They also gave experimental results of the study and obtained that neutrosophic data classifier has a promising tool for data clustering and image processing. Taş et al [29] offered a very practical G-metric based method for data analysis of neutrosophic big data although neutrosophic data type (neutrosophic big data) are in massive and detailed form when compared with other data types. Long et al [30] proposed a new fuzzy clustering algorithm based on association matrix using the neutrosophic set. Basha et al and Kavitha et al [31, 32, 33] gave neutrosophic based prediction system and classifier.

3. Neutrosophic IF-THEN Rules

   Sunay et al [34] used Neutrosophic logic using IF-THEN rules in the risk/safety assessment for the first time. The study focused on linear trapezoidal neutrosophic number. The study of Basha et al [31, 32] identified with "IF-THEN" rules whose antecedents and consequences are composed of neutrosophic logic statements, instead of fuzzy logic ones.

According to [31], NIED includes three functions as the follow:

- Neutrosophication: construction of the neutrosophic knowledge-base by converting crisp inputs using the neutrosophic three membership functions: truth-membership, falsity-membership, and indeterminacy-membership.
- Inference Engine: The KB and neutrosophic "IF-THEN" rules are applied to get a neutrosophic output.
- De-neutrosophication: Converts the neutrosophic output of the previous step back to a crisp value using three functions analogous to the ones used by the neutrosophication.

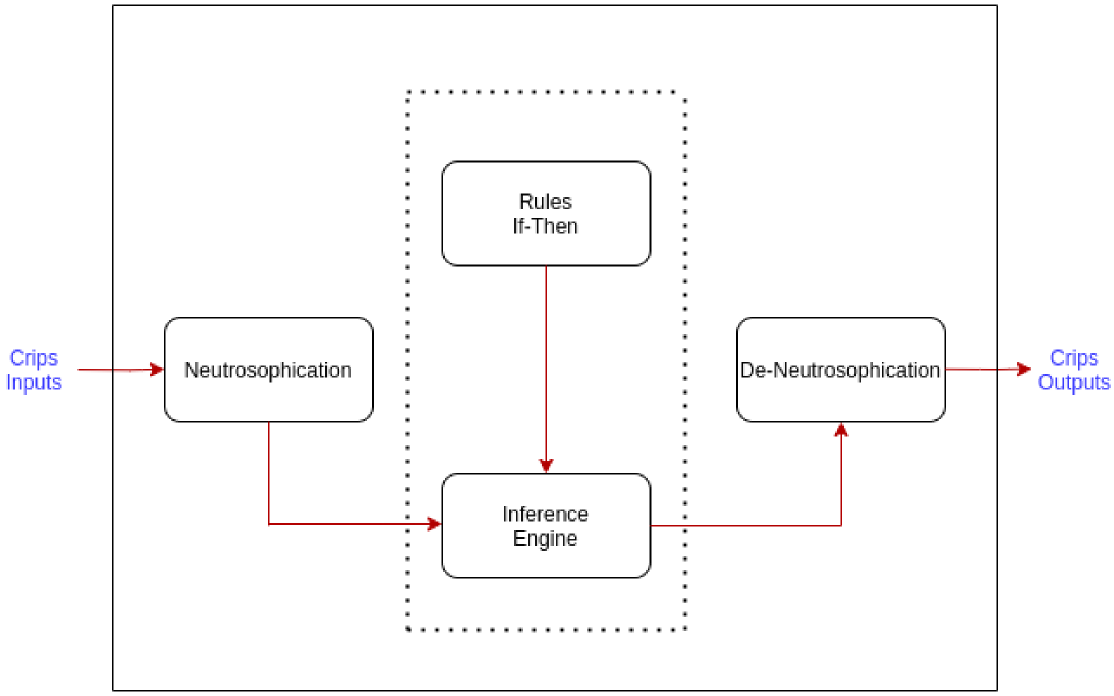

Figure 1: A flowchart for NIED mechanism

We now present our smart agriculture mechanism model integrated with the NIED system in order to monitor the functioning of the ecosystem after both irrigation and seeding in greenhouse or open field. The environment where we want the main process of cabinet to take place is controlled by the control unit. The control unit part is the part that has processors and has automation. Various sensors are available in the cabinet. These sensors make instant readings and transmit the data to the control unit. Since the system will work with electricity, it is particularly sensitive to electrical voltage fluctuations. If this happens, the system is bypassed. UPS is required to prevent this. This is provided by the voltage sensor. Sensors in the cabinet must work continuously. If one of them works abnormally, the system is bypassed. For this reason, it must be an air conditioner system. For example, if the humidity has decreased or the temperature has increased,

the air conditioner system has to intervene. All of these should be related to the control unit and give all the information to it. If it fails, the bypass comes after the off mode and the system shuts down. In order to prevent this situation, the control unit should work in conjunction with the SOS system. When the data becomes abnormal, the control unit intervenes with commands to make the system regular. If the intervention is insufficient, SOS issues a command to the system. The SOS system sends alerts to Mobile Phone, PC etc systems via data network. Here, the security time is 30 minutes. When the control unit detects the error of the system, it activates the SOS system 30 minutes before it closes. NIED system takes place in the control unit and provide a communication decision process with SOS.

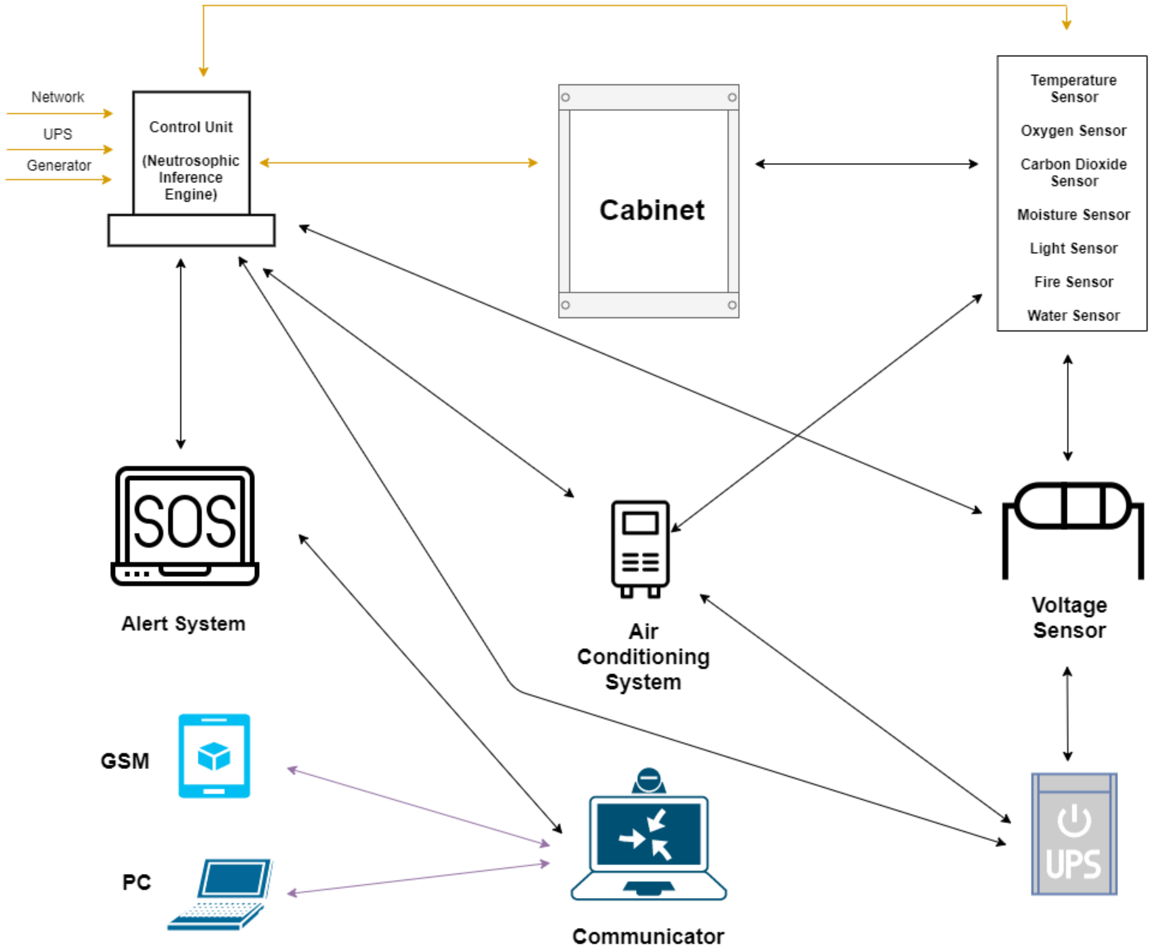

Figure 2: Smart Agriculture Mechanism Model

3.2 Visualization for Soil Surface and Topography

Neutrosophic Bézier surfaces are divided into three distinct form: truth;indeterminancy;falsity Bézier surfaces. Here we introduce new concept of them: geometric mean neutrosophic Bézier surface:

Control points are $GM(x,y) = \sqrt[3]{T(x,y).I(x,y).F(x,y)}$ in z-axis. So its vectorial form is $G(x,y) = (x, y, GM(x,y))$. This is an approximation to the neutrosophic data (see Fig.3). There are several methods to approximate values of data but this one is new in literature. Besides we give intersection curves of the truth-gm, indeterminacy-gm, falsity-gm in figure 4. The advantage of these Bézier curves / surfaces is that they can be written in matrix form. Thus, it can allow neutrosophic data to be matrix form and processed on a computer.

Construction of Bézier surfaces in matrix form is done as follow:

Let $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix}, x = [x_{ij}], V = \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$

where M, U, x, V are the basis matrix of the cubic Bézier surface, matrix of u direction, control points in x-axis and matrix of v direction, respectively. So conversion into polynomial version is matrix multiplication: $U.M.x.M^t.V$. This is done for the y and z axes, respectively. The only thing that changes in this multiplication is the expressions of the control points on the axes.

Neutrosophic If-then rule can be adaptive for this type of visualization. The link between data and visualization is important for the end user. This type of information puts the user-oriented work forward in terms of sales.

Indeed, in this type of study, if it is desired to visualize instantaneous warning boundaries for the user, then the problem of what form the surfaces should be visually arises for what conditions of neutrosophic data and this is definitely an open problem.

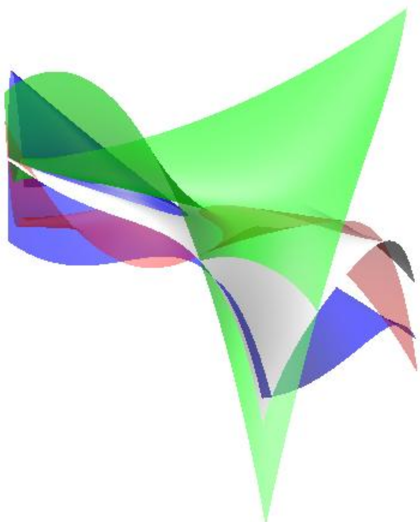

Figure 3: Neutrosophic Bezier surfaces:Truth(blue),indeterminacy(red),falsity(green) and geometric mean surface(gray)

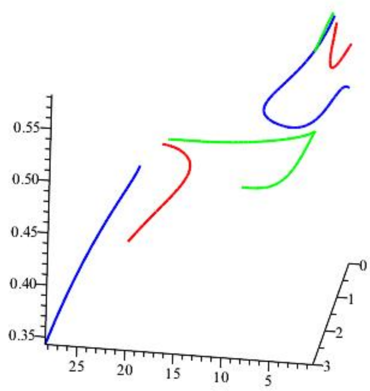

Figure 4: Intersection curves of t-gm(blue),i-gm(red),f-gm(green)

The smart mechanism in Figure 2 scans with the neutrosophic Bezier surface at certain time intervals for both seeding and irrigation of the soil. This scanning process can be included in the neutrosophic inference

engine. With this scanning process, in addition to controlling the soil humidity, temperature, oxygen and carbon dioxide levels, it is checked whether the objects carried to the region by external factors such as wind or animal-human cause damage in the area. With the gray value for Figure 3, it is determined whether there is any change in the soil and on the surface in the scanning study.

4 Conclusion

This study has shown that neutrosophy theory can be used in IoT and Smart Farming systems and at what stages it can help and integrate within the working mechanisms of the systems.Especially, the information coming to the cloud computing or data pool within the systems can make uncertain data process-able with the help of neutrosophic. You can also find structures, models, and researches that can be used to do this processing. On the other hand, two open problems are left that will affect future directions for researchers and neutrosophic-based IoT, Cloud Computing and Smart Agriculture. One of the problems is the de-neutrosophication method for n-gonal neutrosophic numbers and the other is the visualization of the neutrosophic data using the geometric approach.

Conflicts of Interest: The authors declare no conflict of interest.

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